Gradient, Jacobian and Newton-Gauss Hessian

$$E(x_1, x_2, x_3) = \sum_{i=1}^{3} f_i^2(x_1, x_2, x_3)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \qquad g(x) = \frac{dE}{dx} = \begin{bmatrix} \frac{dE}{dx_1} \\ \frac{dE}{dx_2} \\ \frac{dE}{dx_3} \end{bmatrix}$$

$$E(x) = \sum_{i=1}^{3} f_i^2(x)$$

- Expand $f_i(x + \Delta x)$ to a linear form at x
- Let $f_i(x + \Delta x) \approx f_i(x) + \phi_i^T(x)\Delta x$, where $\phi_i(x) = \dfrac{df_i(x)}{dx}$

$$J(x_1, x_2, x_3) = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{bmatrix} = \begin{bmatrix} \phi_1^T(x) \\ \phi_2^T(x) \\ \phi_3^T(x) \end{bmatrix}$$

$$E(x) = \sum_{i=1}^{3} f_i^2(x)$$

$$E(x + \Delta x) = \sum_{i=1}^{3} f_i^2(x + \Delta x) \approx \sum_{i=1}^{3} (f_i(x) + \phi_i^T(x)\Delta x)^2$$

$$E(x+\Delta x) = \sum_{i=1}^{3} f_i^2(x+\Delta x) \approx \sum_{i=1}^{3} (f_i(x) + \phi_i^T(x)\Delta x)^2$$

$$= \sum_{i=1}^{3} (f_i^2(x) + 2f_i(x)\phi_i^T(x)\Delta x) + (\phi_i^T(x)\Delta x)^2)$$

$$= \sum_{i=1}^{3} f_i^2(x) + 2\sum_{i=1}^{3} f_i(x)\phi_i^T(x)\Delta x + \sum_{i=1}^{3} (\phi_i^T(x)\Delta x)^2$$

$$E(x + \Delta x)$$

$$\approx \sum_{i=1}^{3} f_i^2(x) + 2\sum_{i=1}^{3} f_i(x)\phi_i^T(x)\Delta x + \sum_{i=1}^{3} (\phi_i^T(x)\Delta x)^2$$

$$E(x) = \sum_{i=1}^{3} f_i^2(x)$$

$$2\sum_{i=1}^{3} f_i(x)\phi_i^T(x)\Delta x$$

$$= (2\sum_{i=1}^{3} f_i(x)\phi_i^T(x))\Delta x$$

$$\sum_{i=1}^{3} (\phi_i^T(x)\Delta x)^2 = \sum_{i=1}^{3} (\phi_i^T(x)\Delta x)^T(\phi_i^T(x)\Delta x)$$

$$= \sum_{i=1}^{3} (\Delta x^T \phi_i(x))(\phi_i^T(x)\Delta x)$$

$$= \Delta x^T(\sum_{i=1}^{3} \phi_i(x)\phi_i^T(x))\Delta x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad F(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} \qquad J(x_1, x_2, x_3) = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{bmatrix} = \begin{bmatrix} \phi_1^T(x) \\ \phi_2^T(x) \\ \phi_3^T(x) \end{bmatrix}$$

$$2 \sum_{i=1}^{3} f_i(x) \phi_i^T(x) \Delta x$$

$$= 2 \left[f_1(x), f_2(x), f_3(x) \right] \begin{bmatrix} \phi_1^T(x) \\ \phi_2^T(x) \\ \phi_3^T(x) \end{bmatrix} \Delta x$$

$$= 2F^T(x) J(x) \Delta x$$

$$E(x+\Delta x) \approx L(x+\Delta x) = E(x) + g^T(x)\Delta x + \frac{1}{2}\Delta x^T H \Delta x$$

$$g^T(x) = 2F^T(x)J(x)$$

$$g(x) = 2J^T(x)F(x)$$

$$H(x) = 2\sum_{i=1}^{3} \phi_i(x)\phi_i^T(x)$$