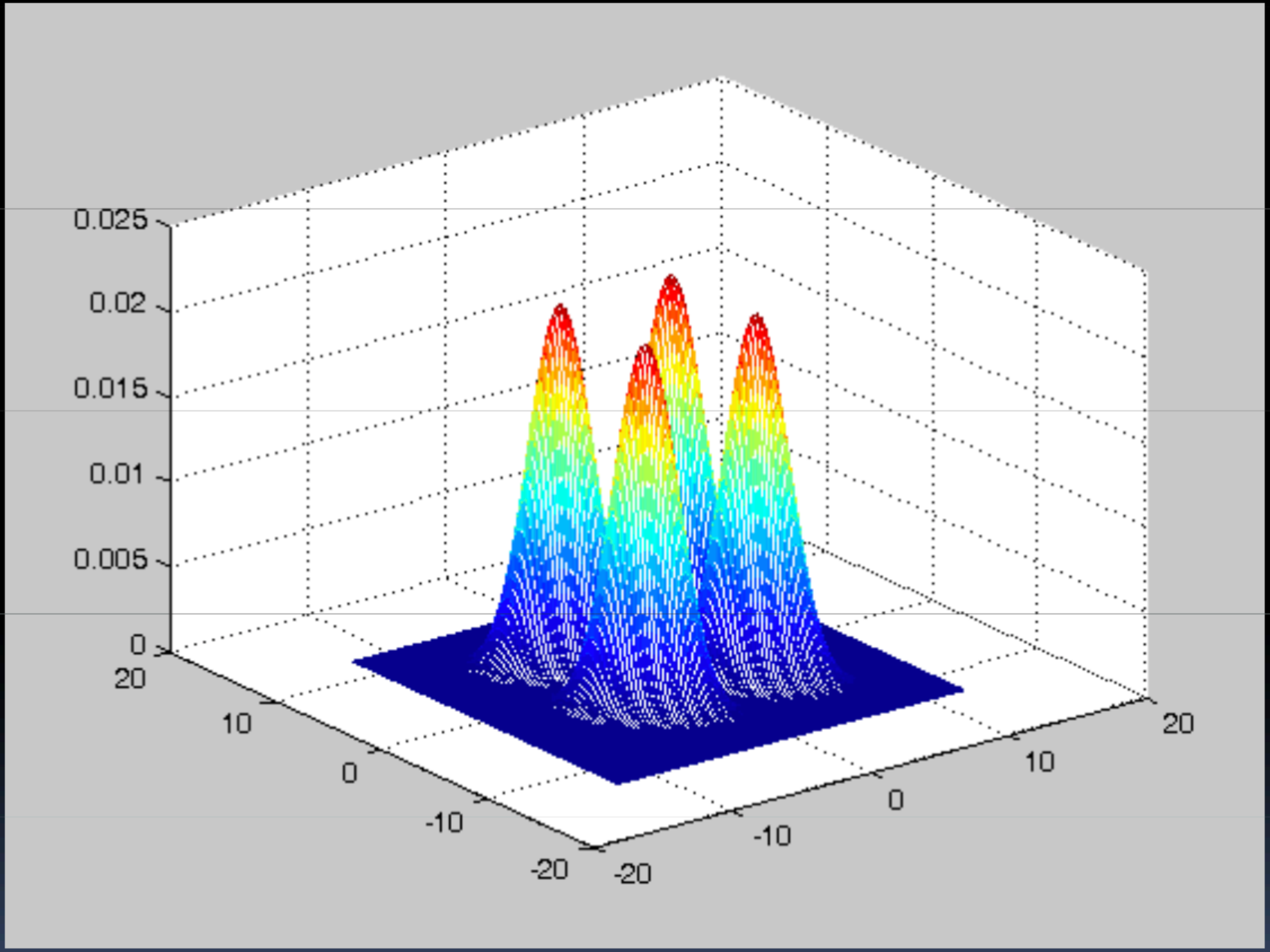
Numerical Integration

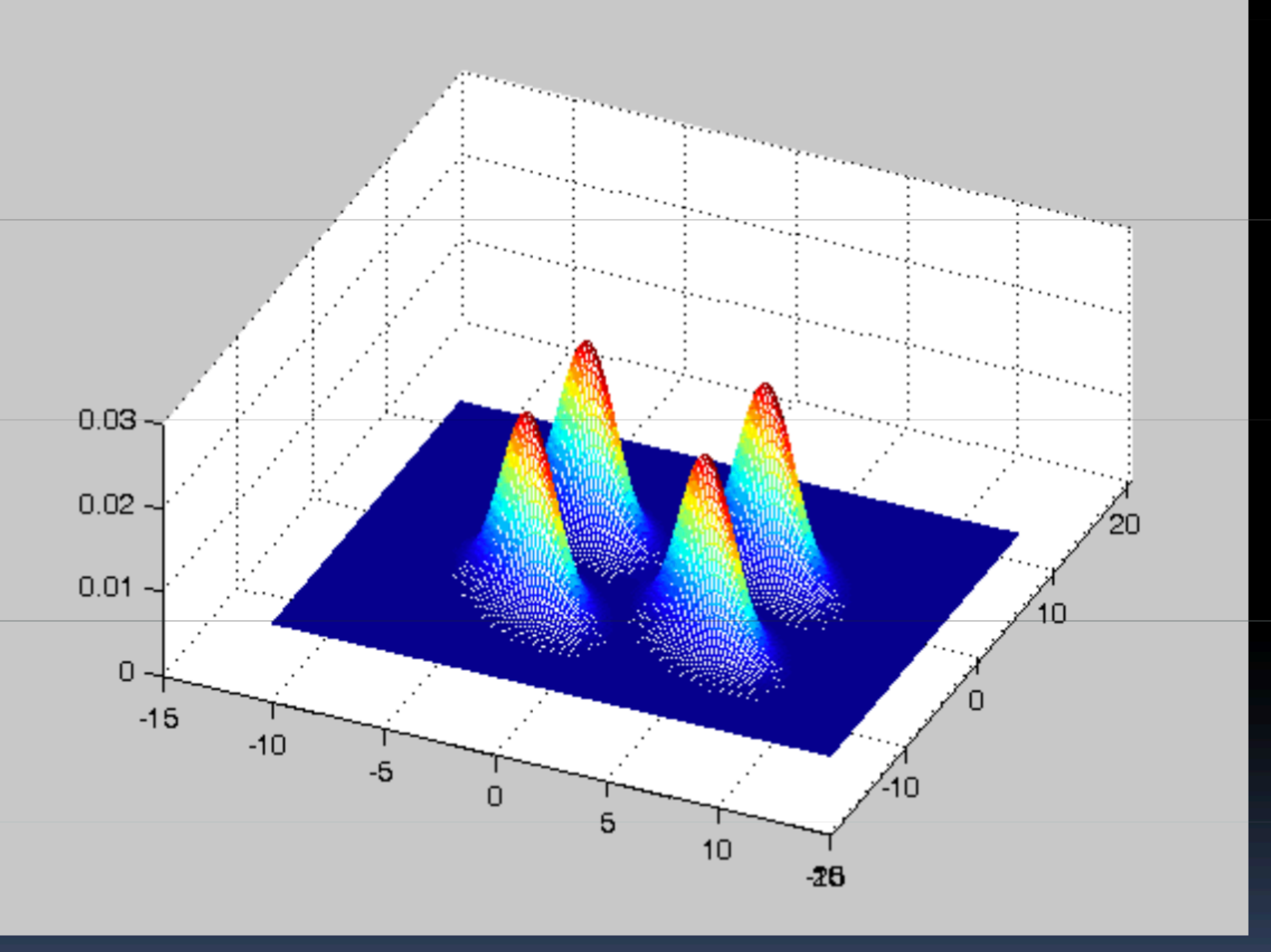
合成梯形法 Composite Trapzoid

合成新普森法 Composite Simpson

Four 2-variate Gaussians



Four 2-variate Gaussians



Gaussian pdf

$$P_k(x) = P(x|y_k, A_k)$$

$$= \frac{1}{(2\pi)^{d/2}\sqrt{|A_k^{-1}|}} \exp\left(\frac{(x-y_k)^t A_k (x-y_k)}{2} \right)$$

Weight sum of Gaussian pdfs

$$q(\mathbf{x}) = \sum_k \pi_k p_k(\mathbf{x} \mid \mathbf{A}, \mathbf{y}_k)$$

myfx4.m

```
function out=myfx4(x,y)
global ep;
    A=[0.8 0.2; 0.3 0.75]; d=4;
    u(1,:)=[d d]; u(2,:)=[-d d];
    u(3,:)=[d -d]; u(4,:)=[-d -d];
    A=A'*A;n=length(x);
    c=1/(2*pi*sqrt(det(inv(A))));
    tx=[x;ones(size(x))*y]';
    for i=1:n
        out(i)=0;
         for j=1:4
            out(i)=out(i)+1/4*c*exp(-(tx(i,:)-u(j,:))*A*(tx(i,:)-u(j,:))'/2);
         end
    end
return
```

Plot 4G

plot_4G.m

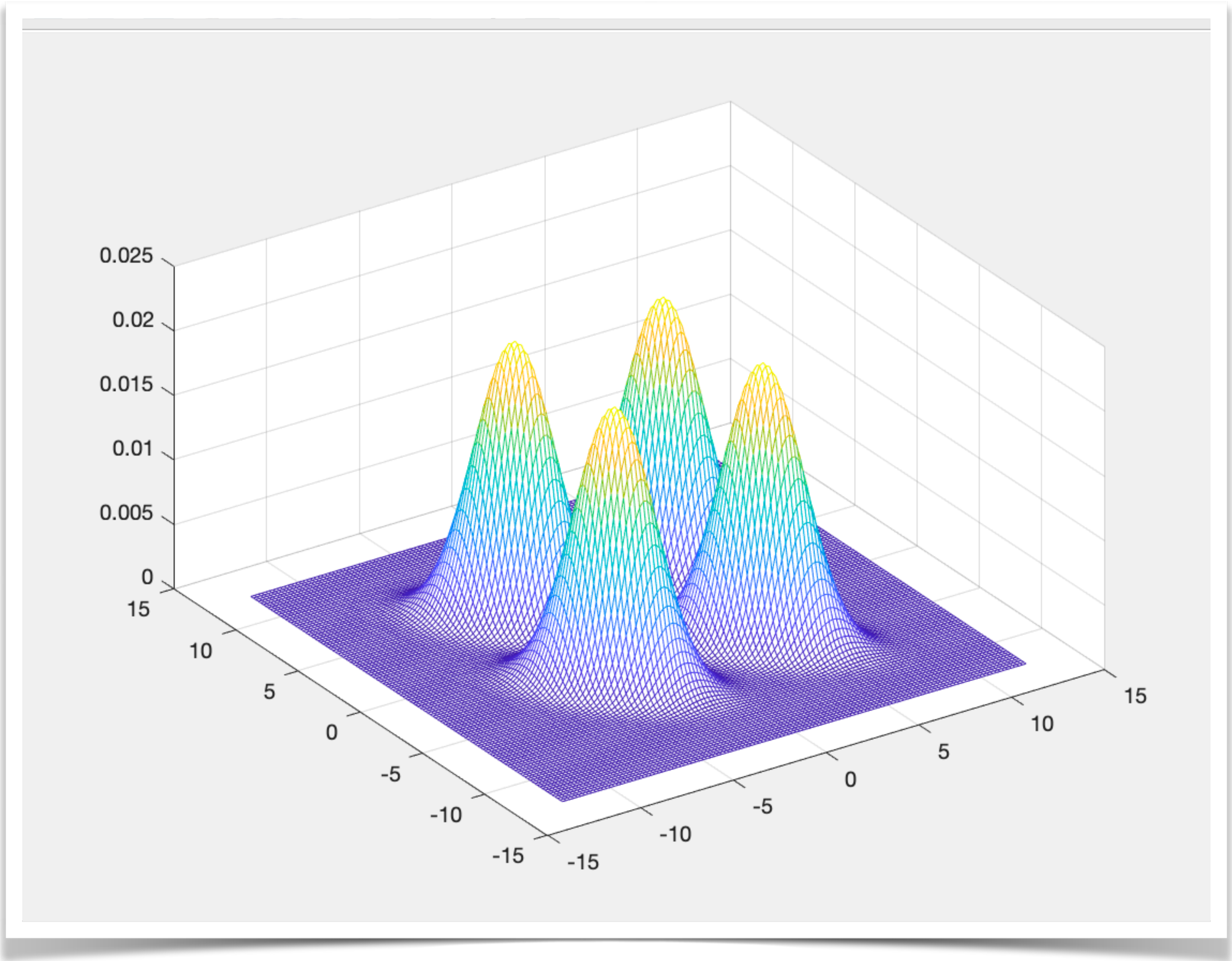
```
function plot_4G
n=4;
range=n*pi;
x1=-range:0.2:range;
x2=x1;
for i=1:length(x1)
        y_hat=myfx4(x2,x1(i));
        C(i,:)=y_hat;
end
fprintf('max value of fx4:%f\n',max(max(C)));
mesh(x1,x2,C);
```

Integration of 4G

demo_int_4G.m

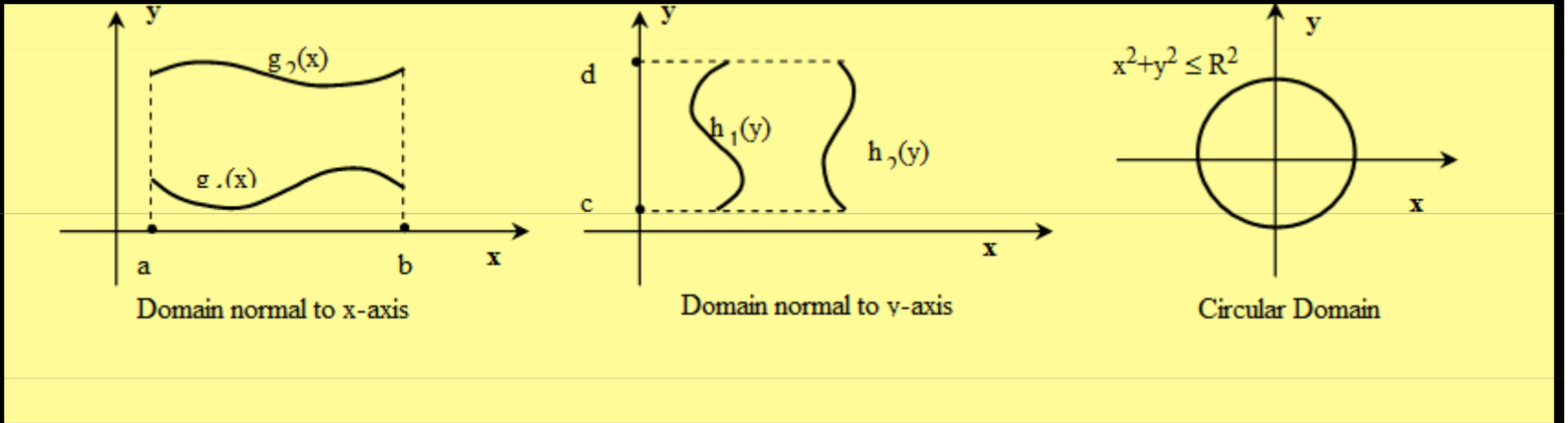
Double integration →

```
function demo_int_4G()
n=4;
xmin = -n*pi;
xmax = n*pi;
ymin = -n*pi;
ymax = n*pi;
result = dblquad(@myfx4,xmin,xmax,ymin,ymax);
fprintf('integration of fx4 over the region :%f \n',result);
```




```
>> demo_int_4G
integration of fx4 over the region :1.000002
```

Numeric calculus for double integrals



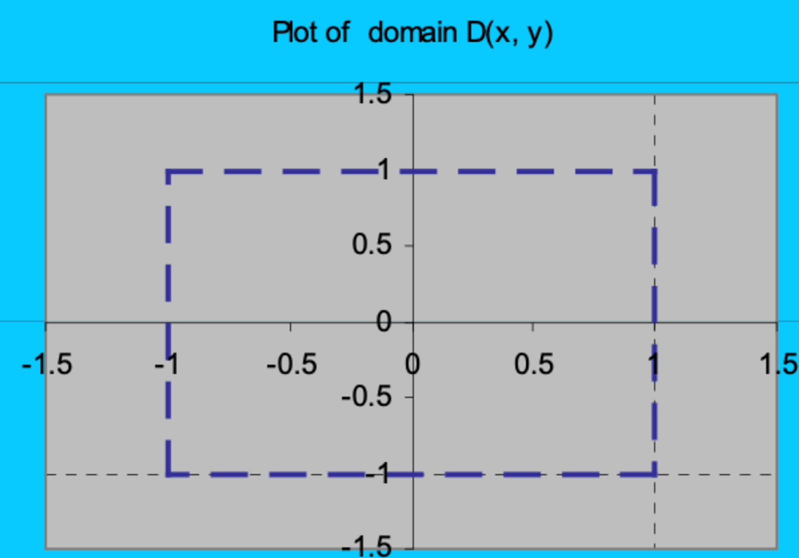
$$\iint_{D_X} f(x,y)ds = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

$$\iint_{D_Y} f(x,y)ds = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

$$\iint_C f(x,y)ds = \int_0^{2\pi} \int_0^R f(\rho\cos(\theta), \rho\sin(\theta))\rho \, d\rho \, d\theta$$

Example demo_ex2_4G.m

$$\iint_D f(x,y)ds = \int_{-1}^{1}\int_{-1}^{1} e^{-(x^2+y^2)}\, dy\, dx \qquad \pi \cdot (erf(1))^2$$



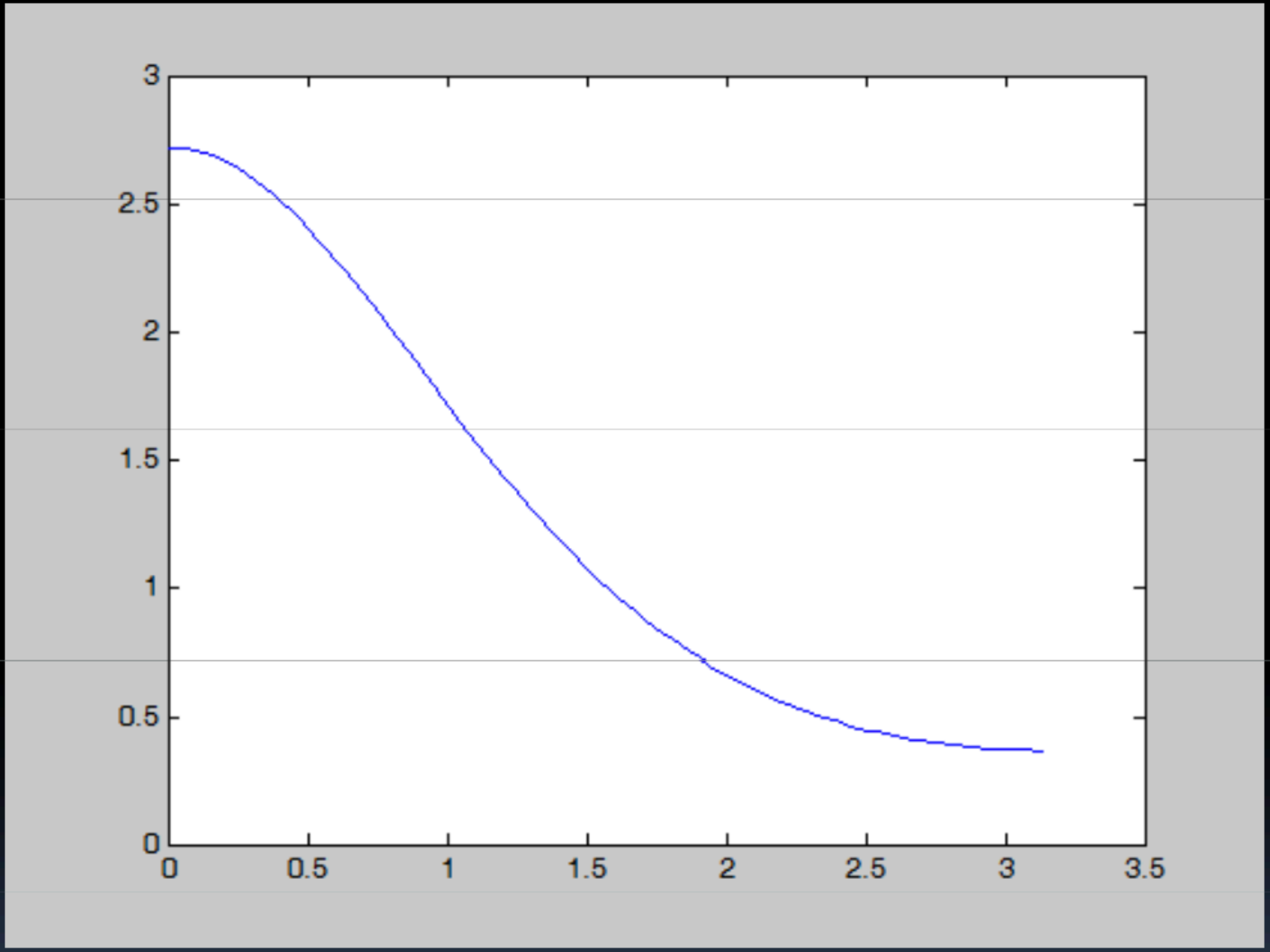
Approx. integral	2.230985141404140
True Integral	2.230985141404130

```
>> demo_exG2
integration of fG2 over the region :2.2309851725856071347
```

exp(cos(x))

plot_expcos.m

```
function plot_expcos()
x=linspace(0,pi);
plot(x,fx(x));

function y=fx(x)
    y=exp(cos(x));
return
```



Definite Integration

demo_quad.m

```
function demo_quad()
q = quad(@fx,0,pi);
fprintf('quadrature = %f\n',q);

function y=fx(x)
    y=exp(cos(x));
return
```

```
>> demo_quad
quadrature = 3.977463
```

Symbolic integration

demo_int.m

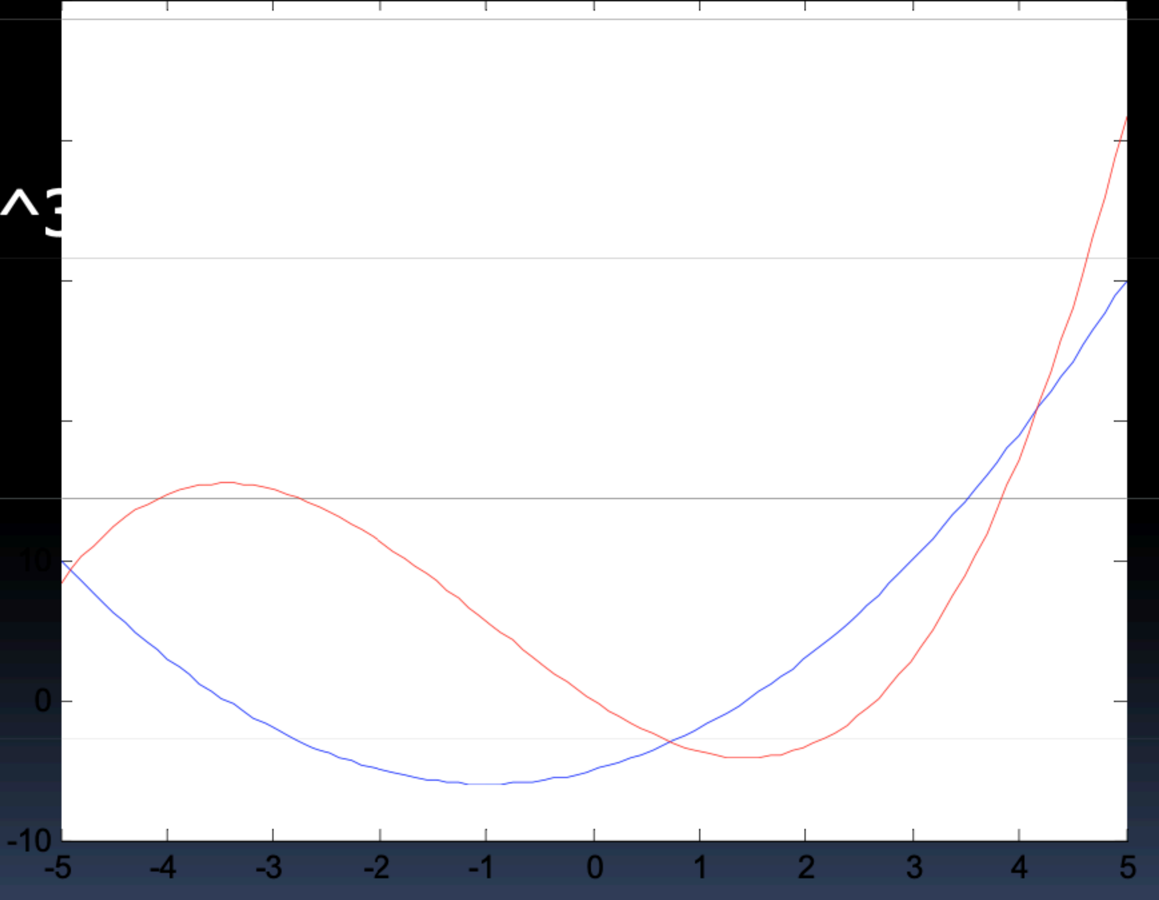
```
ss=input('function of x:','s');
fx=inline(ss);
x=sym('x');
ss="int(" + ss + ")";
ss1=eval(ss);
```

Example

function of x:x.^2+2*x-5

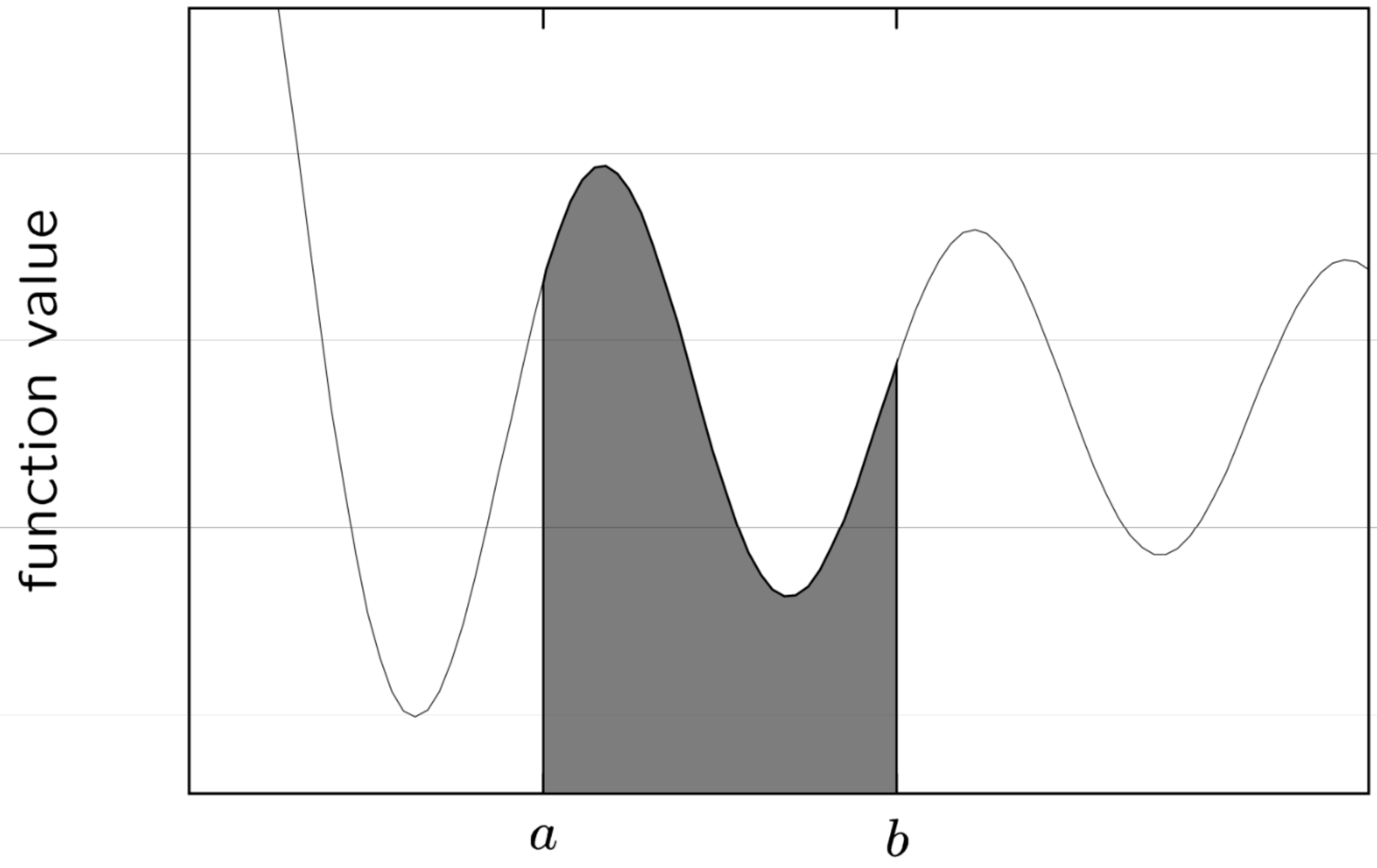
fx1 =

Inline function:
fx1(x) = 1./3.*x.^3



Numerical integration - quadrature

$f(x) \geq 0$ on $[a, b]$ bounded $\Rightarrow \int_a^b f(x)\,dx$ is area under $f(x)$



Counter example

$$\int_0^{\frac{\pi}{2}} \left[1 - a^2 \sin^2 \theta\right]^{\frac{1}{3}} d\theta$$

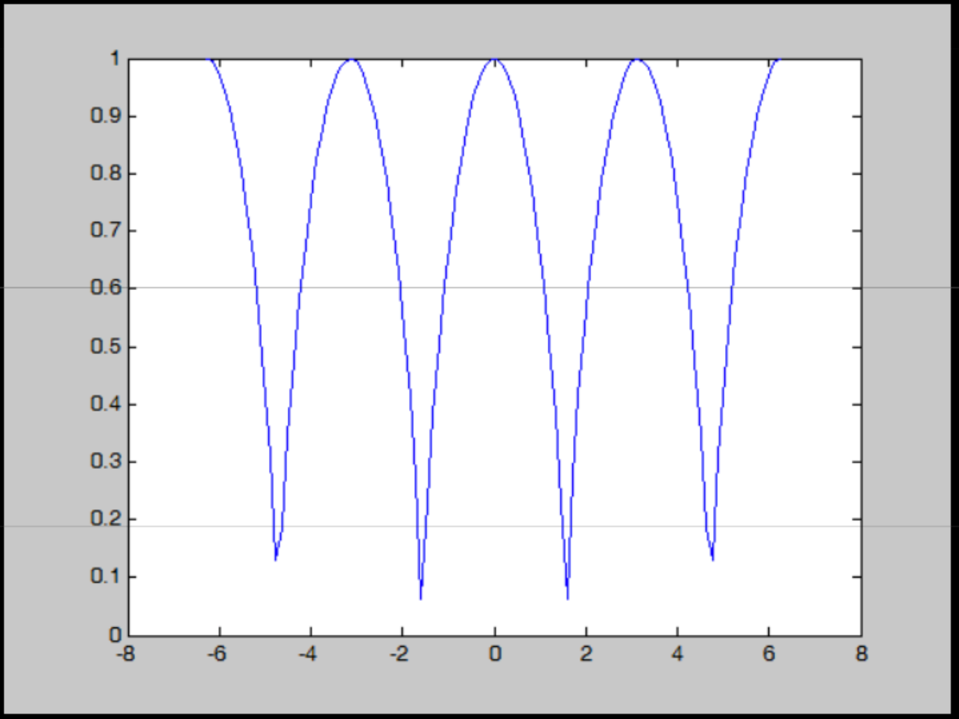
```
>> demo_int
function of x:(1-sin(x.^2)).^(1/3)
Warning: Explicit integral could not be found.
```

```
ss=input('function of x:','s');
fx=inline(ss);
x=sym('x');
ss="int(" + ss + ")";
ss1=eval(ss)
```

```
function of x:(1-sin(x^2))^(1/3)

ss1 =

int((1 - sin(x^2))^(1/3), x)
```

plot_sin13.m

```
ss='(1-sin(x).^2).^(1/3)'
fx=inline(ss);
x=linspace(-2*pi,2*pi);
plot(x,fx(x))
```



Numerical integration

```
function demo_quad2()
q = quad(@fx,0,pi);
fprintf('quadrature = %f\n',q);

function y=fx(x)
y = (1-sin(x).^2).^(1/3);
```

```
>> demo_quad2
quadrature = 2.240498
```

Mesh

$$P \equiv \{a = x_0 < x_1 < \cdots < x_n = b\}$$

Infima and suprema:

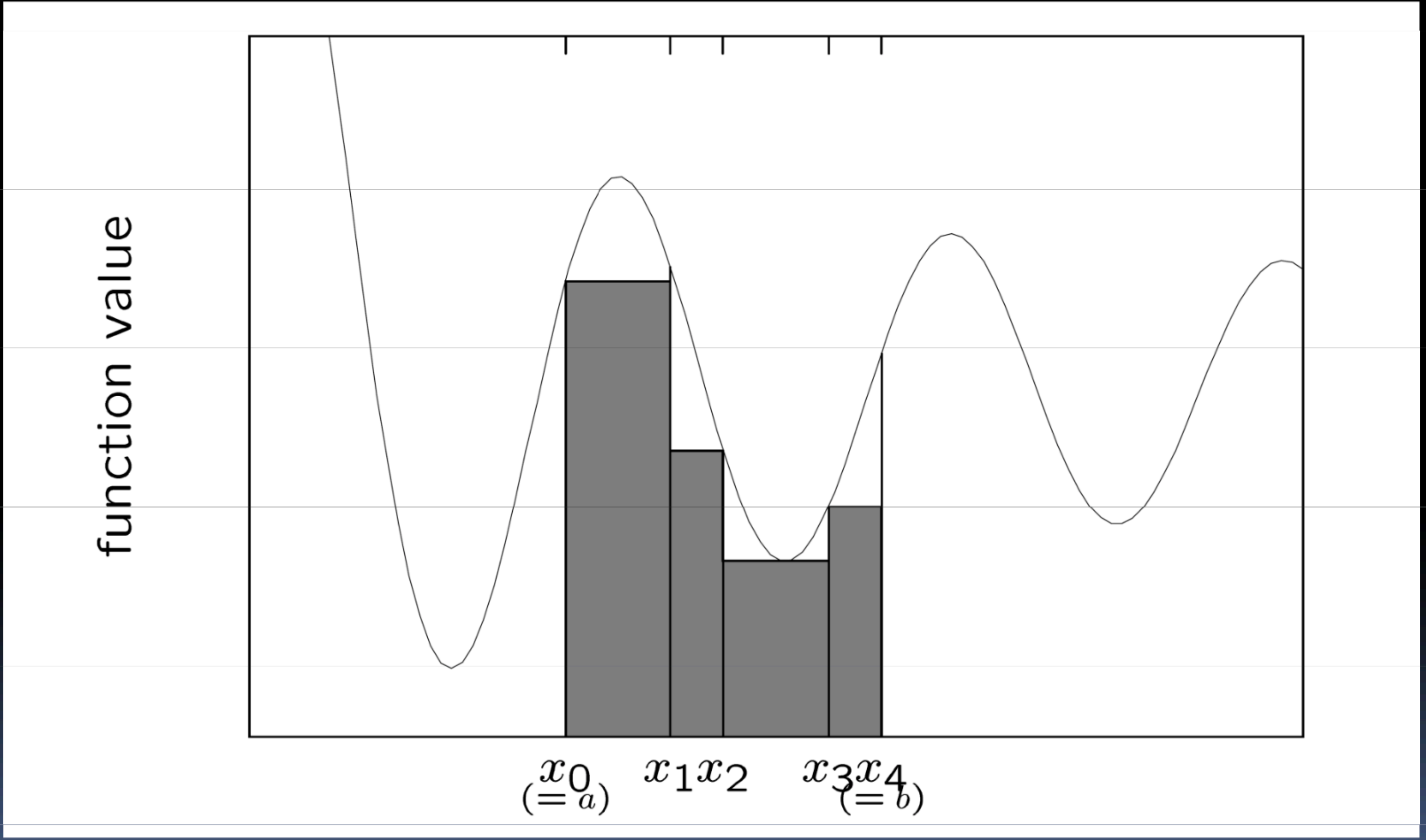
$$m_i \equiv \inf \left\{ f(x) : x_i \le x \le x_{i+1} \right\}$$
$$M_i \equiv \sup \left\{ f(x) : x_i \le x \le x_{i+1} \right\}$$

Lower and upper sum

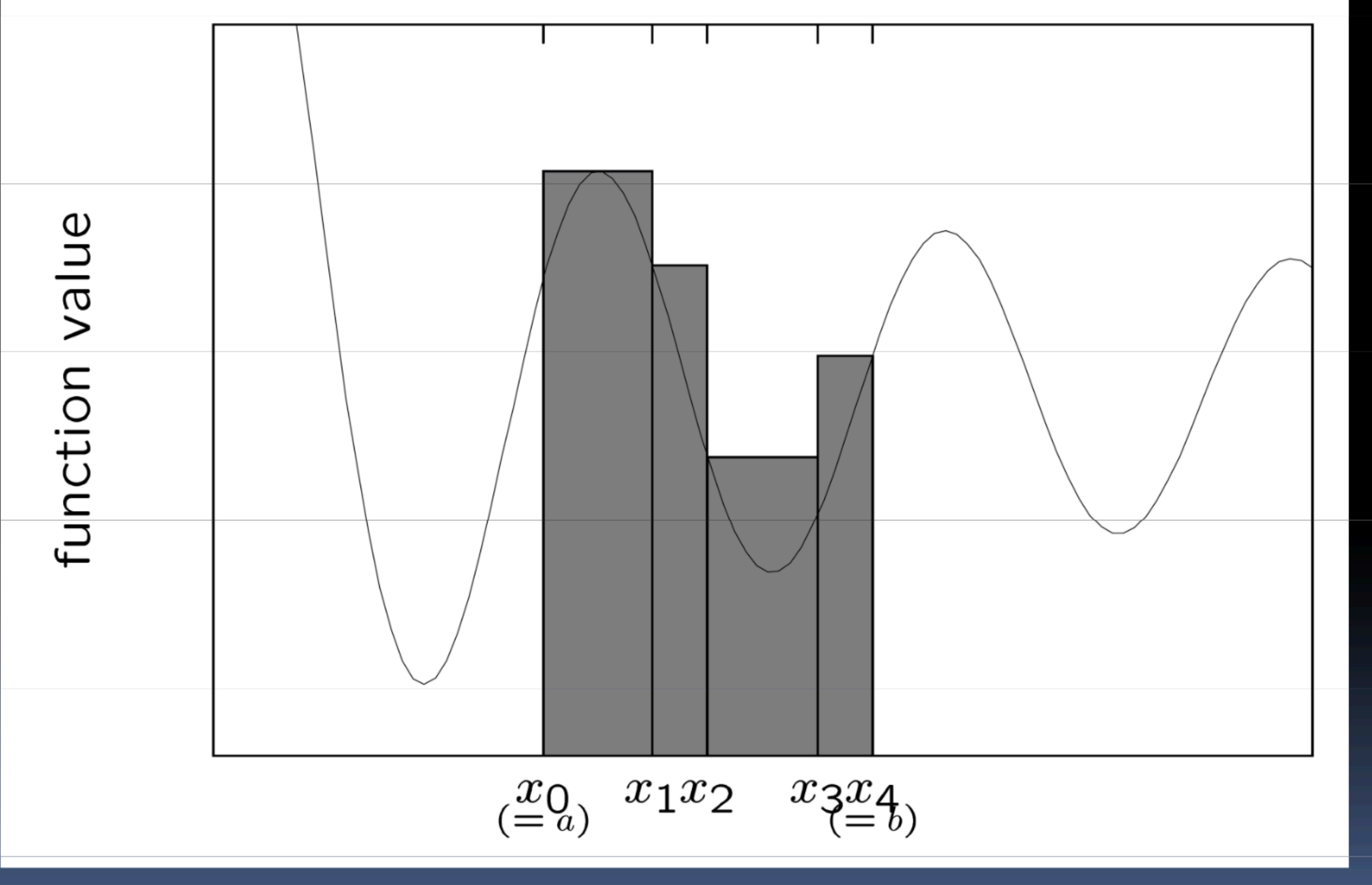
$$L(f;P) \equiv \sum_{i=0}^{n-1} m_i \left(x_{i+1} - x_i\right)$$

$$U(f;P) \equiv \sum_{i=0}^{n-1} M_i \left(x_{i+1} - x_i\right)$$

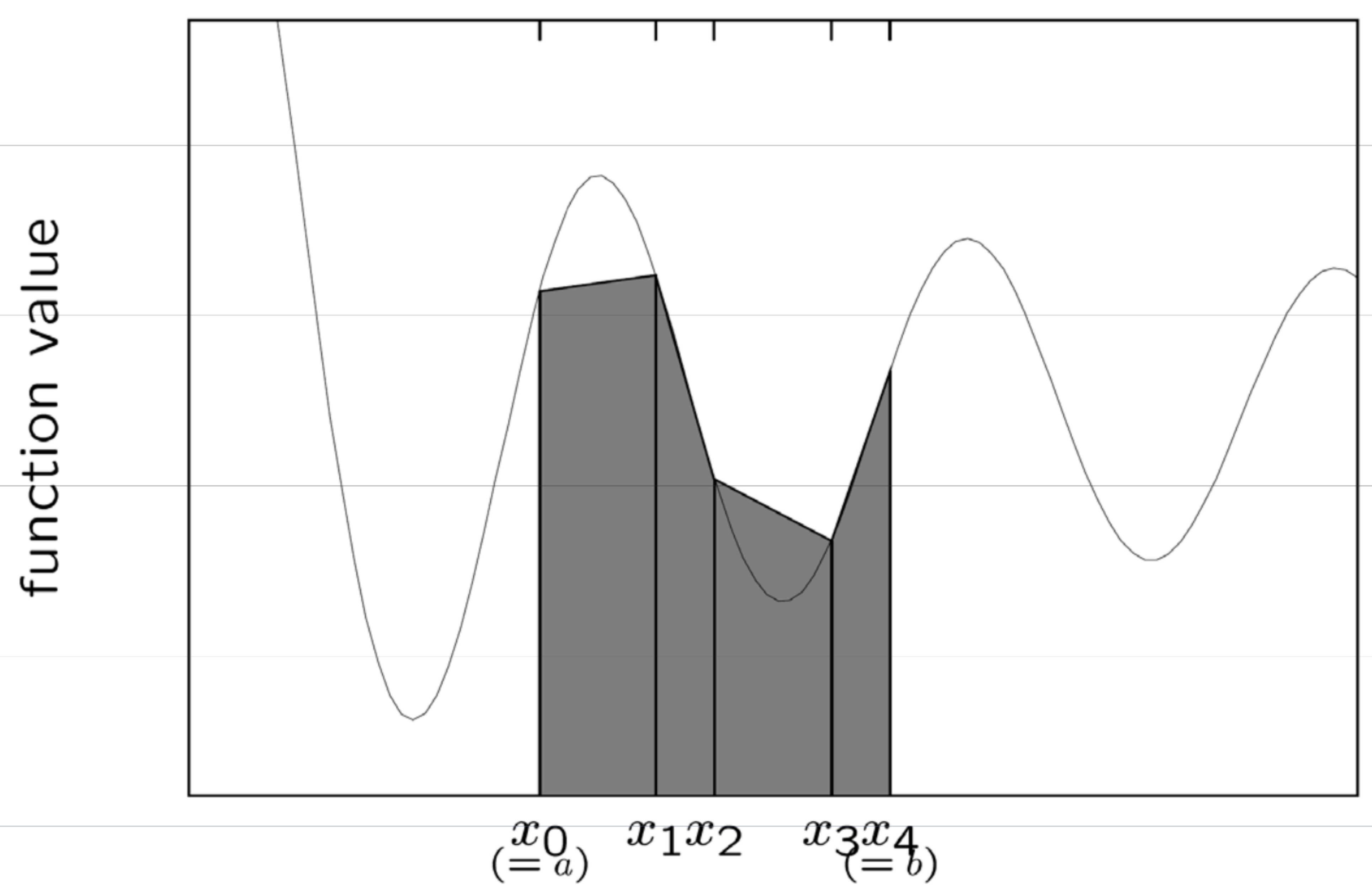
Lower sum : lower bound



Upper sum : upper bound



Composite Trapezoid rule



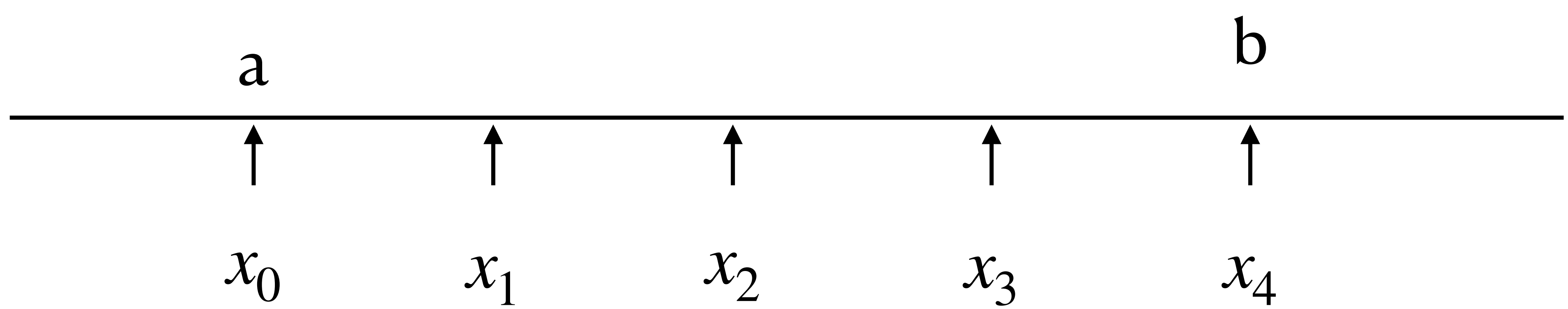
Uniform mesh

Constant stepsize $h = \frac{b-a}{n}$

$$T(f;P) \equiv h\left\{\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}[f(x_0) + f(x_n)]\right\}$$

Constant stepsize $h = \frac{b-a}{n}$

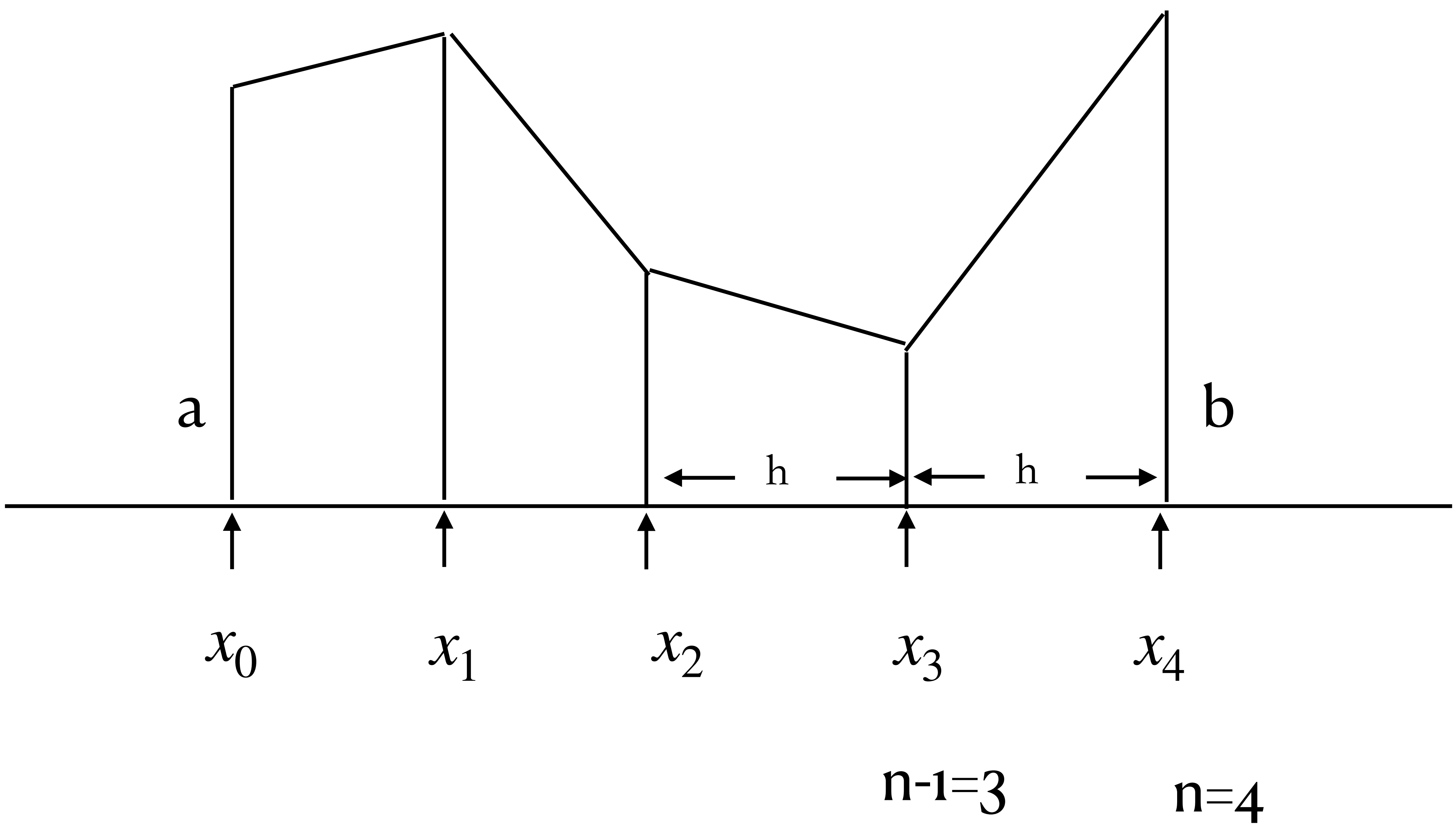
$$T(f;P) \equiv h\left\{\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}[f(x_0) + f(x_n)]\right\}$$



n=4

Constant stepsize $h = \frac{b-a}{n}$

$$T(f;P) \equiv h\left\{\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}[f(x_0) + f(x_n)]\right\}$$



Error Analysis

Theorem: $f \in C^2[a,b] \rightarrow \exists \xi \in (a,b) \ni$

$$\int_a^b f(x)\,dx - T(f;P) = -\frac{1}{12}(b-a)h^2 f''(\xi) = O\left(h^2\right)$$

Partition size

$$\int_0^\pi e^{\cos x}\,dx,\ \text{error tolerance} < \tfrac{1}{2}\times 10^{-3},\ n = ?$$

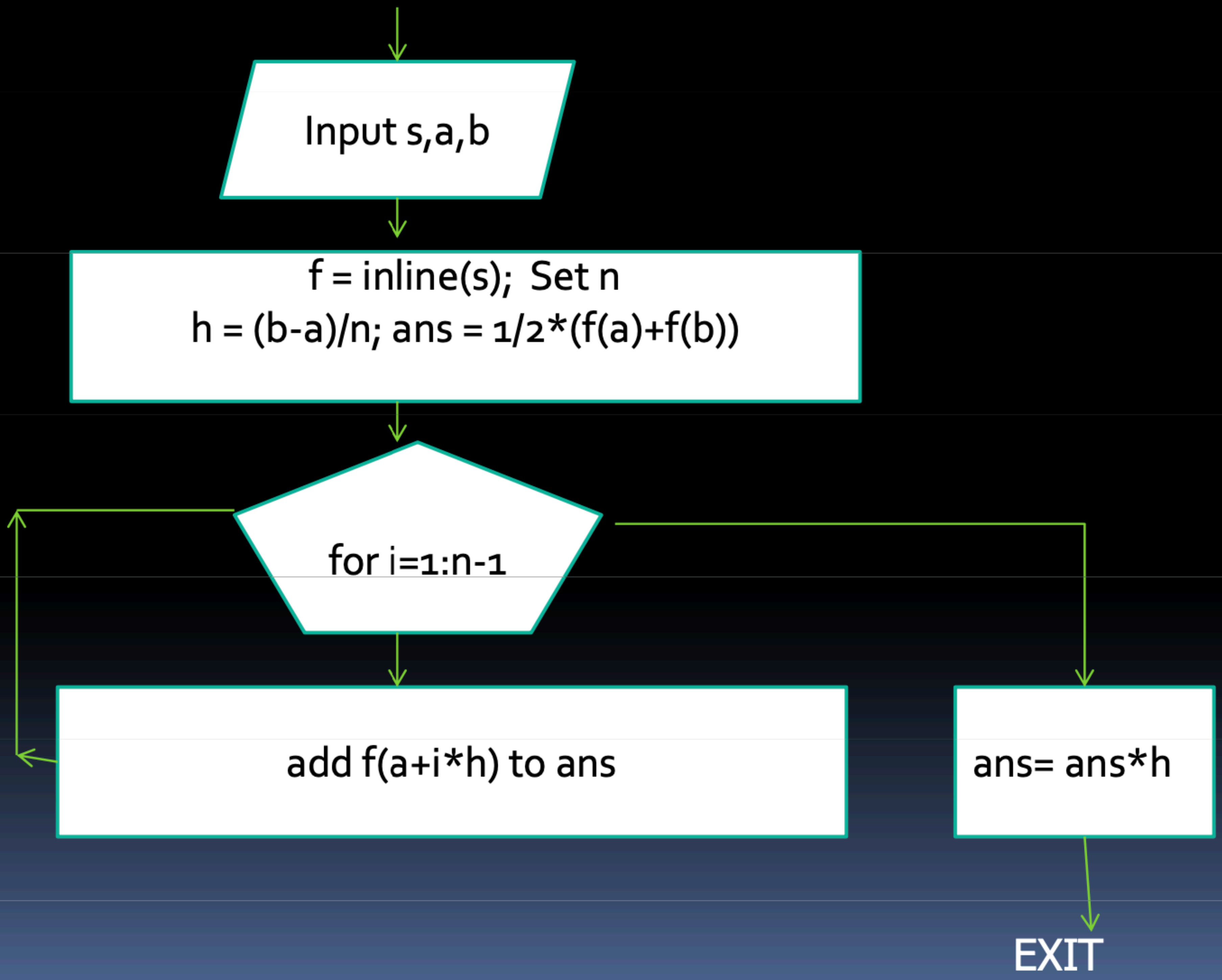
- $f(x) = e^{\cos x} \Rightarrow f'(x) = -e^{\cos x}\sin x \ \ldots\ |f''(x)| \le e$ on $[0,\pi]$
- $\therefore\ |\text{error}| < \frac{1}{12}\pi(\pi/n)^2 e < \frac{1}{2}\times 10^{-3}$
- $\ldots\ n \ge 119$

Composite Trapezoid rule

- input s, a and b
- f = inline(s); Set n
- h = (b-a)/n; ans = 1/2*(f(a)+f(b))
- for i=1:n-1

  add f(a+i*h) to ans
- ans= ans*h

Flow Chart


Input s,a,b
f = inline(s); Set n
h = (b-a)/n; ans = 1/2*(f(a)+f(b))
for i=1:n-1
add f(a+i*h) to ans
ans= ans*h
EXIT

Simpson rule for numerical integration

If $f \in C^4[a,b]$, then a number ξ in (a, b) exists with

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]-\frac{f^{(4)}(\xi)}{2880}(b-a)^5$$

Exercise

- Draw a flow chart to illustrate integration by the composite Trapezoid rule
- Implement the composite Trapezoid rule for numerical integration, including flow chart and Matlab codes
- Test your matlab function with the following integration

$$\int_0^{\pi} f(x)$$

f(x)=exp(cos(x))

- * Test your matlab function with definite integration of the weight sum of four Gaussian pdfs
- * Compare your results with those obtained by using quad.m

Simpson rule
Composite Simpson rule

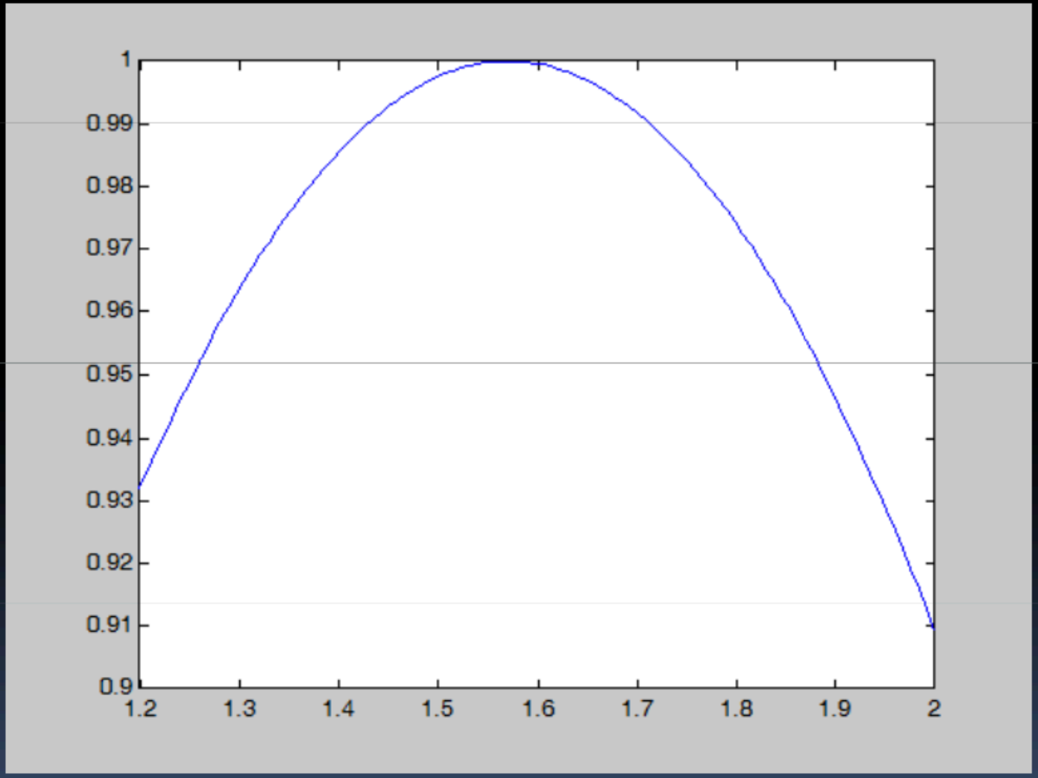
Simpson rule for numerical integration

If $f \in C^4[a,b]$, then a number ξ in (a, b) exists with

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]-\frac{f^{(4)}(\xi)}{2880}(b-a)^5$$

y=sin(x)   for x within [1.2 2]

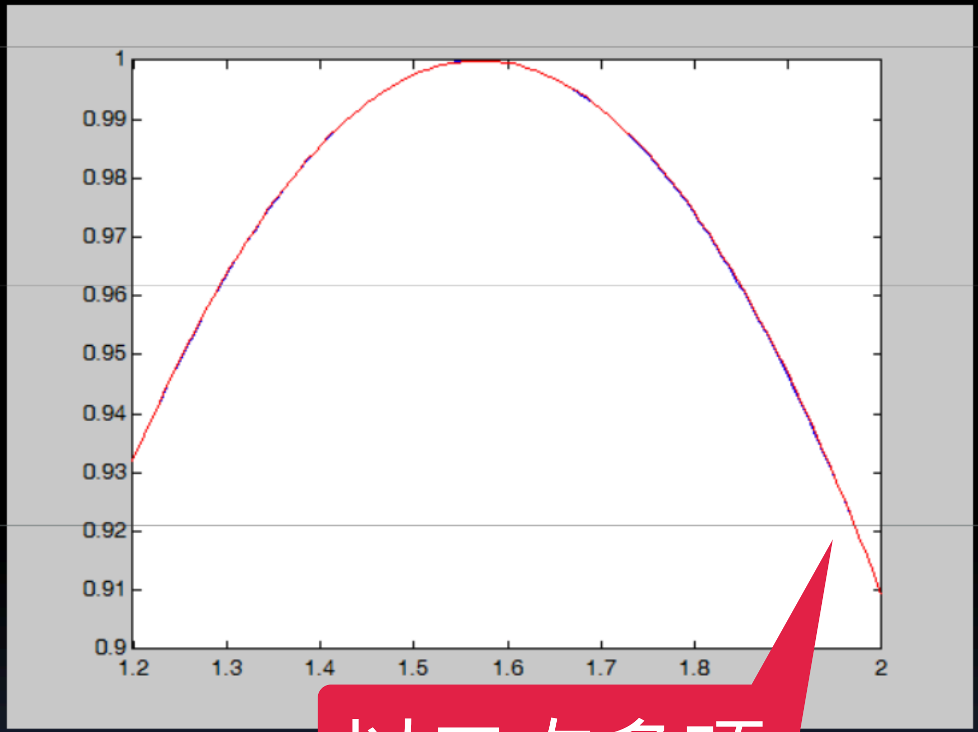
a=1.2;b=2;
x=linspace(a,b);plot(x,sin(x));hold on



Quadratic polynomial

- Approximate sin using a quadratic polynomial

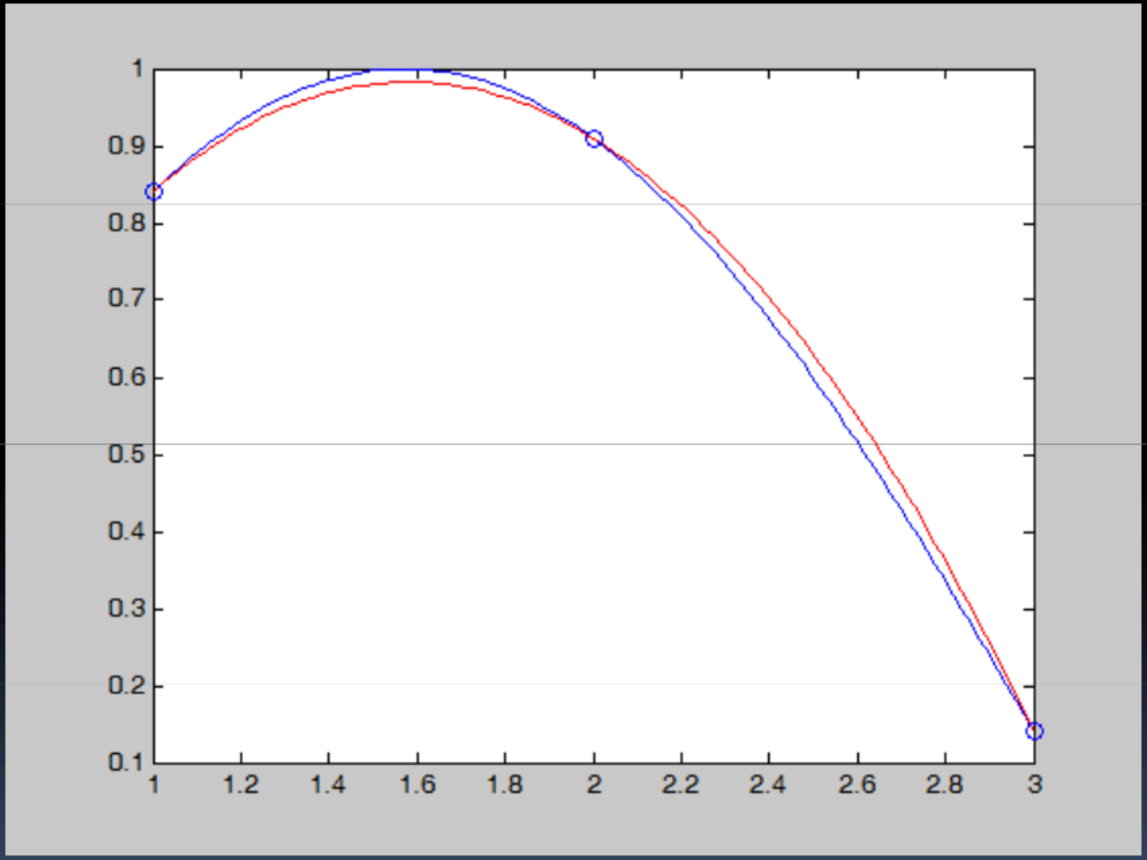
```
c=0.5*(a+b);
x=[a b c]; y=sin(x);
p=polyfit(x,y,2);
z=linspace(a,b);
plot(z,polyval(p,z),'r')
```



以二次多項式近似sin

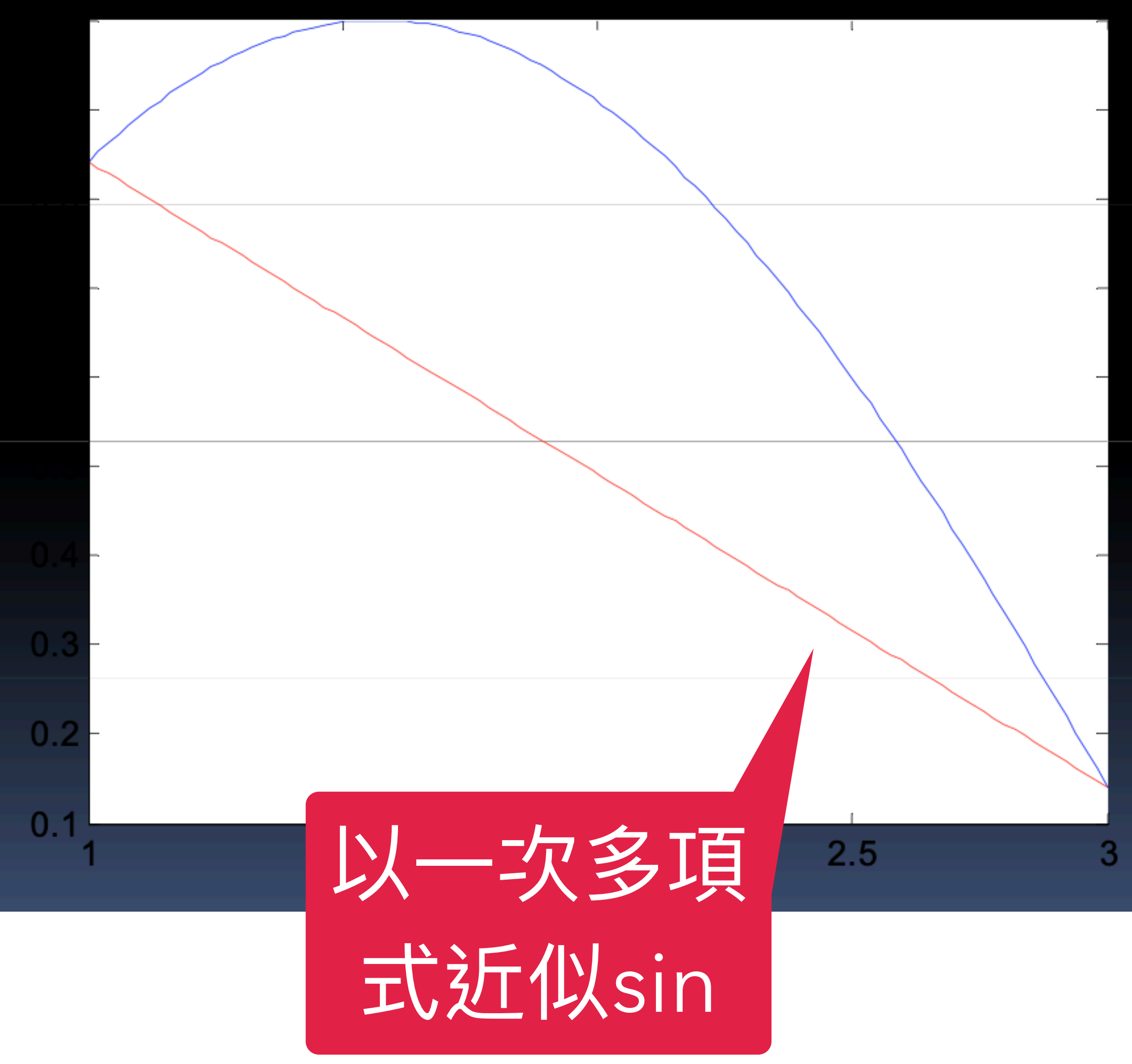
Approximate sin(x) within [1 3] by a quadratic polynomial

```
a=1;b=3;
x=linspace(a,b);plot(x,sin(x))
hold on;
c=0.5*(a+b);
x=[a b c]; y=sin(x);
p=polyfit(x,y,2);
z=linspace(a,b);

plot(z,polyval(p,z) ,'r')
```



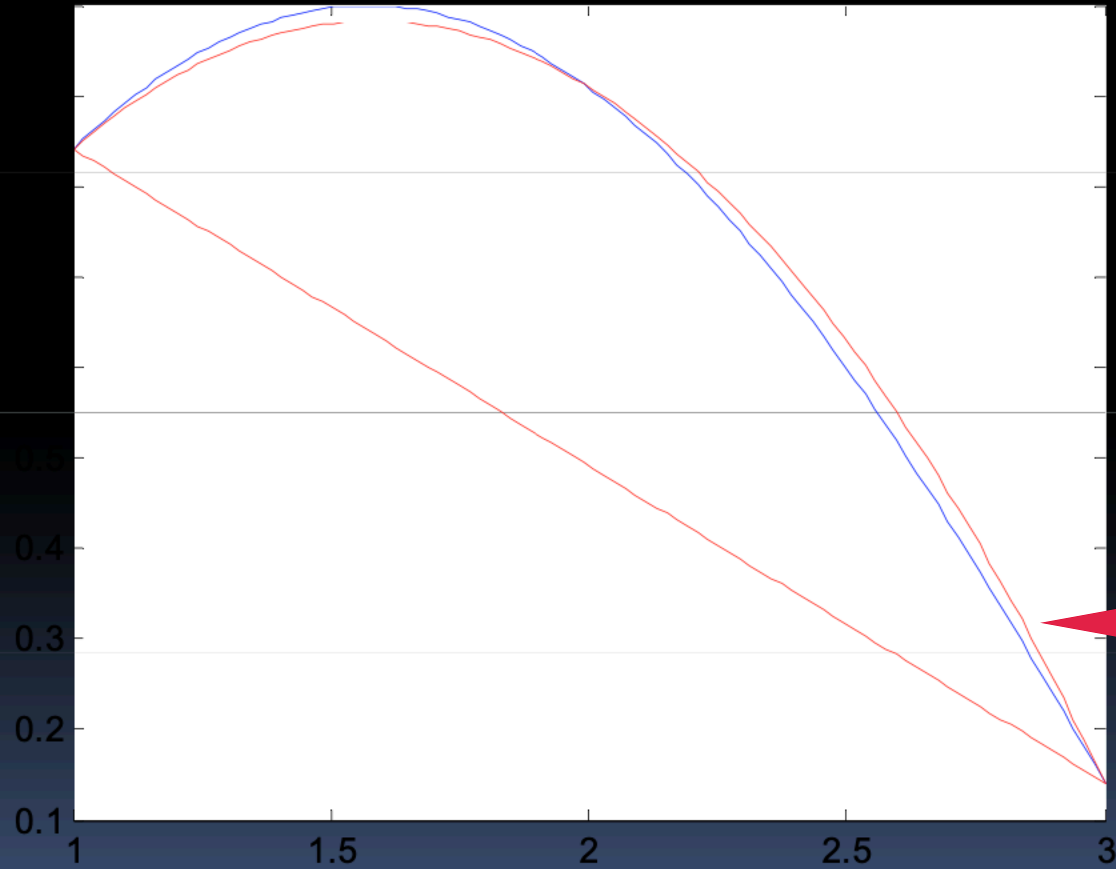
Approximate sin(x) within [1 3] by a line

```
a=1;b=3;
x=linspace(a,b);plot(x,sin(x))
hold on;
c=0.5*(a+b);
x=[a b ]; y=sin(x);
p=polyfit(x,y,1);
z=linspace(a,b);

plot(z,polyval(p,z) ,'r')
```



Strategy I : Apply Trapezoid rule to calculate area under a line
Strategy II : Apply Simpson rule to calculate area under a second order polynomial

Strategy II is more accurate and general than strategy I, since a line is a special case of quadratic polynomial

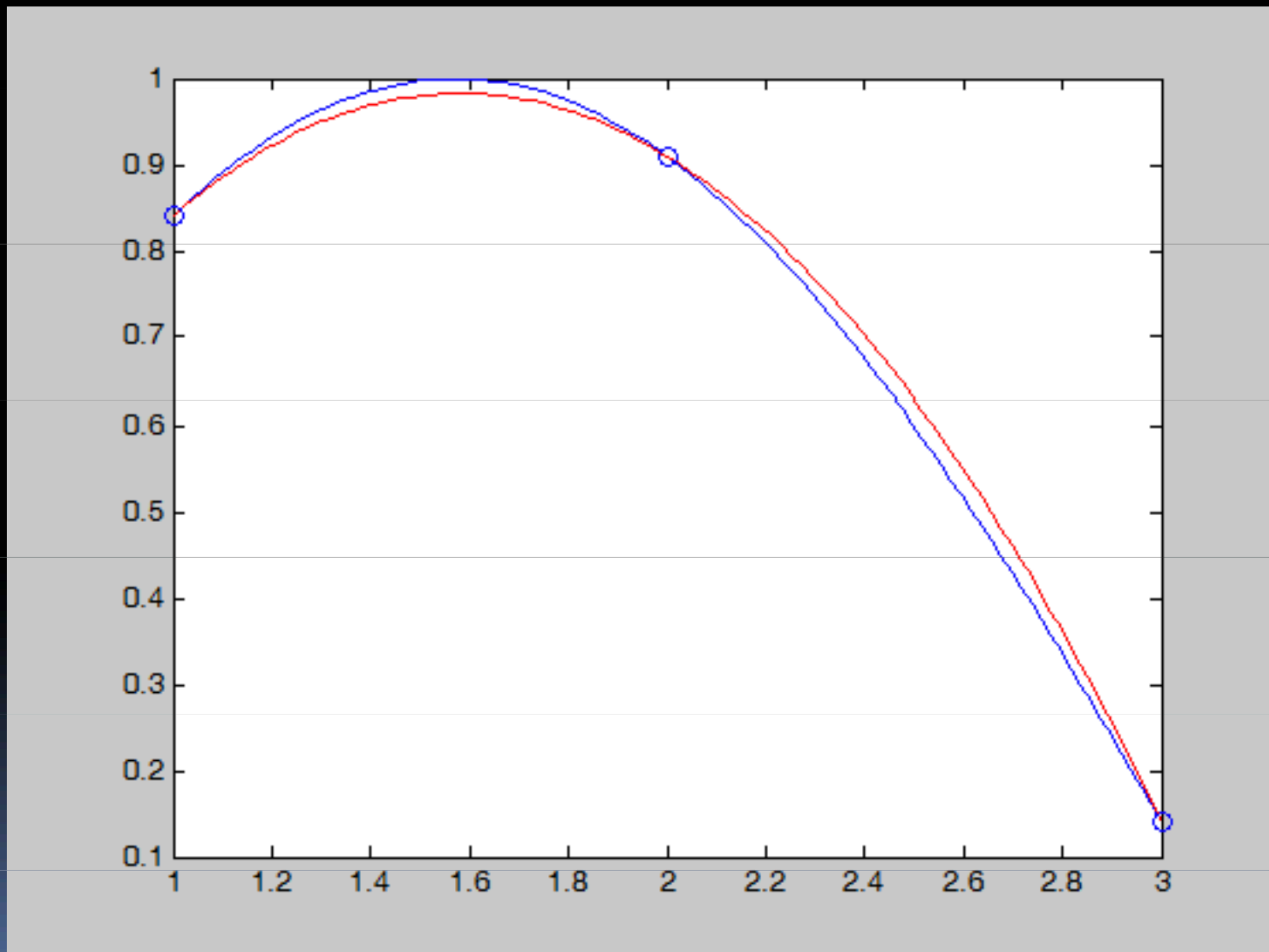


以二次多項式近似比較精確，誤差較小

Simpson rule

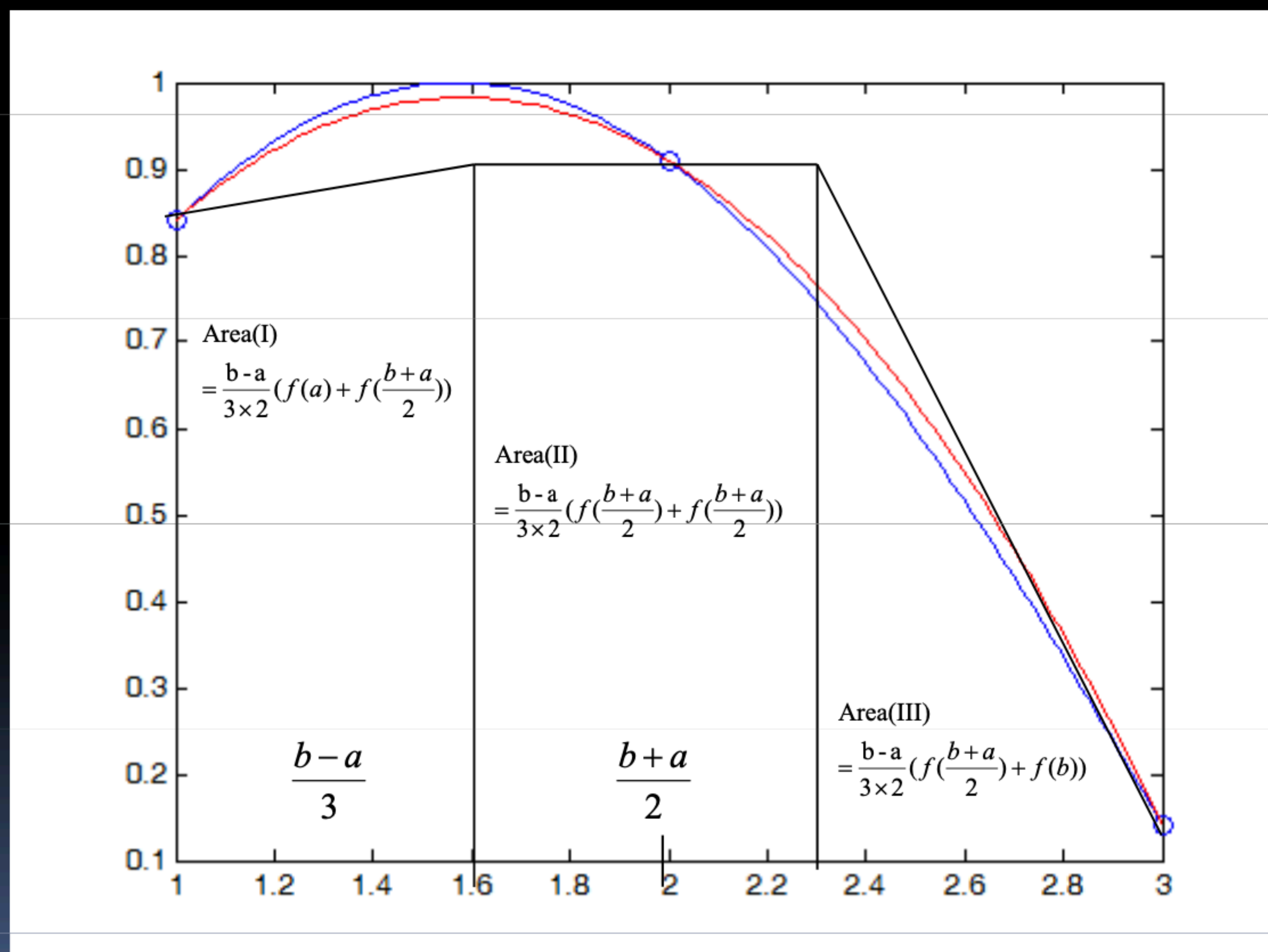
- Original task: integration of f(x) within [a,b]
- c=(a+b)/2
- Numerical task
  - Approximate f(x) within [a,b] by a quadratic polynomial, p(x)
  - Integration of p(x) within [a,b]

Blue: f(x)=sin(x) within [1,3]
Red: a quadratic polynomial that pass (1,sin(1)), (2,sin(2)),(3,sin(3))



- The area under a quadratic polynomial is a sum of area I, II and III

通過點 $(a, f(a)), (\frac{a+b}{2}, f(\frac{a+b}{2})), (b, f(b))$的二次多項式與橫軸間的面積



Area(I)
+Area(II)
+Area(III)

- The area under a quadratic polynomial is a sum of area I, II and III

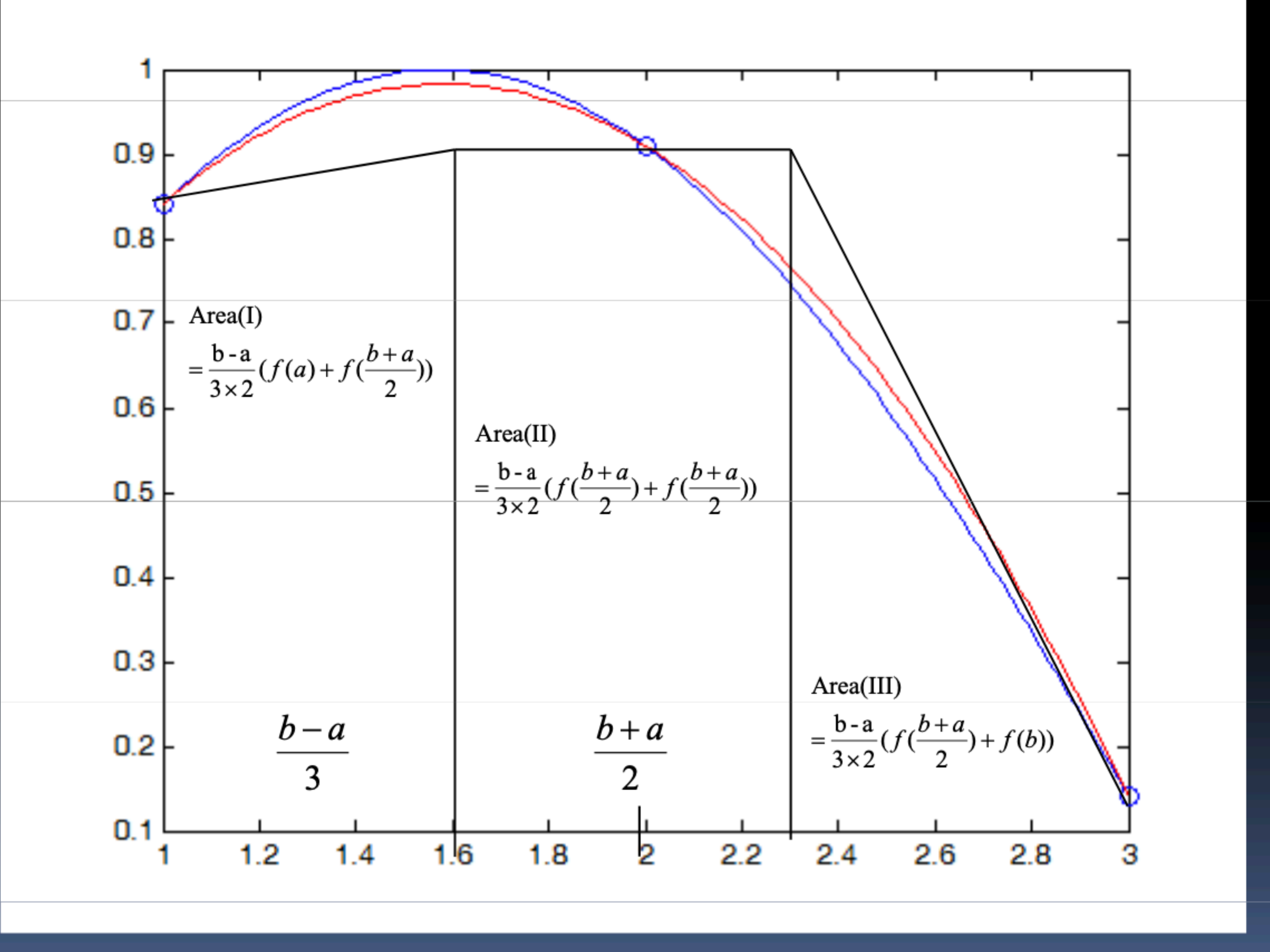
- Partition [a,b] to three equal-size intervals, and use the high of the middle point c to produce three Trapezoids

- Use the composite Trapezoid rule to determine area I, II and III
  h/2*(f(a)+f(c) +f(c)+f(c)+ f(c)+f(b))

- substitute h=(b-a)/3   and c=(a+b)/2
- area I + II + III = (b-a)/6 *(f(a)+4*f((a+b)/2) +f(b))

$$\text{Area(I)} + \text{Area(II)} + \text{Area(III)}$$

$$= \frac{\text{b - a}}{3 \times 2}\left(f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right)$$



$p(x) \in P_2$

$p(x)$ passes $(a, f(a))$ $(b, f(b))$ $(\frac{a+b}{2}, f(\frac{a+b}{2}))$

Show that

$$\int_a^b p(x)dx = \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right]$$

$$(a, f(a))$$

$$(b, f(b))$$

$$(c, f(c)), c = \frac{a+b}{2}$$

$$p(x) = \left[f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)} \right]$$

$$proof : p(a) = f(a), p(b) = f(b), p(c) = f(c)$$

$$\int_a^b p(x)dx = \int_a^b \left[f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)} \right] dx$$

$$= f(a)\int_a^b \frac{(x-b)(x-c)}{(a-b)(a-c)}dx + f(b)\int_a^b \frac{(x-a)(x-c)}{(b-a)(b-c)}dx + f(c)\int_a^b \frac{(x-a)(x-b)}{(c-a)(c-b)}dx$$

$b \rightarrow a+2h$

$c \rightarrow a+h$

$$= \frac{h}{3}(f(a) + 4f(c) + f(b))$$

$$= \frac{(b-a)}{6}(f(a) + 4f(c) + f(b))$$

$$\int_a^b \frac{(x-b)(x-c)}{(a-b)(a-c)}dx = \int_a^{a+2h} \frac{(x-a-2h)(x-a-h)}{(a-b)(a-c)}dx = h\int_0^2 \frac{(t-2)(t-1)}{(0-2)(0-1)}dt = h\frac{1}{3}$$

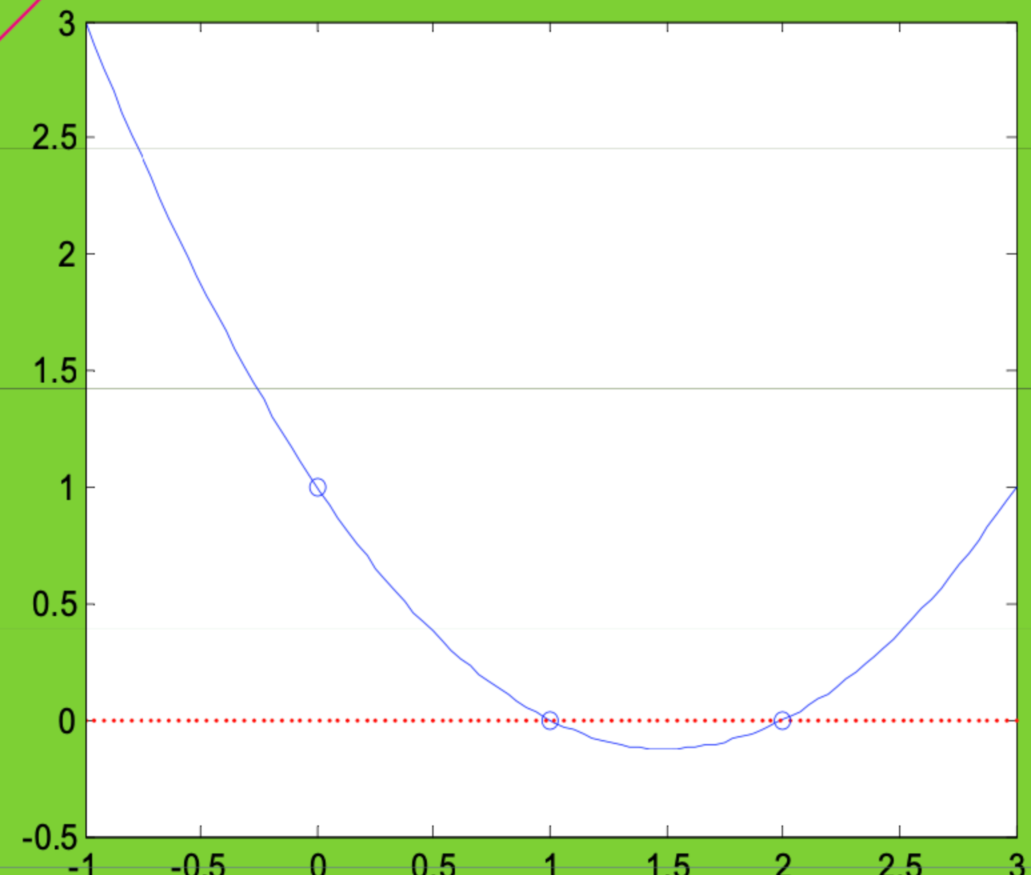
$$\int_a^b \frac{(x-a)(x-b)}{(c-a)(c-b)}dx = h\int_0^2 \frac{(t-0)(t-2)}{(1-2)(1-0)}dt = h\frac{4}{3}$$

$$\int_a^b \frac{(x-a)(x-c)}{(b-a)(b-c)}dx = h\int_0^2 \frac{(t-0)(t-1)}{(2-1)(2-0)}dt = h\frac{1}{3}$$

$$\int_a^b \frac{(x-b)(x-c)}{(a-b)(a-c)}dx = \int_a^{a+2h} \frac{(x-a-2h)(x-a-h)}{(a-b)(a-c)}dx$$

$$= \int_0^{2h} \frac{(t-2h)(t-h)}{(0-2h)(0-h)}dt = h\int_0^{2} \frac{(t-2)(t-0)}{(0-2)(0-1)}dt$$

hift Lagrange polynomial
escale Lagrange polynomial



Shortcut to Simpson 1/3 rule

$$p(x) = Ax^2 + Bx + C$$

$$p \in P_2$$

$$\int_a^b p(x)dx = Q(I) + Q(II) + Q(III)$$

$$= \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$$

1. Partition [a,b] to three equal intervals
2. Draw three trapezoids, Q(I), Q(II) and Q(III)
3. Calculate the sum of areas of the three trapezoids

Proof

$$p(x) = \frac{1}{(a-b)^2}\left[2f(a)(x-b)(x-c) + 2f(b)(x-a)(x-c) - 4f(c)(x-a)(x-b)\right]$$

$$= Ax^2 + Bx + C$$

$$p \in P_2$$

$$\int_a^b p(x)dx = \int_a^b (Ax^2 + Bx + C)dx$$

$$= \left[\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right]_a^b$$

$$= \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$$

Composite Simpson rule

- Partition [a,b] into n interval
- Integrate each interval by Simpson rule
- h=(b-a)/2n

$$\int_a^b f(x)dx$$

$$= \sum_{i=0}^{n-1} \int_{a+2ih}^{a+(2i+2)h} f(x)dx$$

Apply Simpson sule to each interval

$$\int_a^b p(x)dx$$

$$= \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$$

$$\int_{a+2ih}^{a+(2i+2)h} f(x)dx$$

$$= \frac{h}{3}(f(a+2ih) + 4f(a+(2i+1)h) + f(a+(2i+2)h))$$

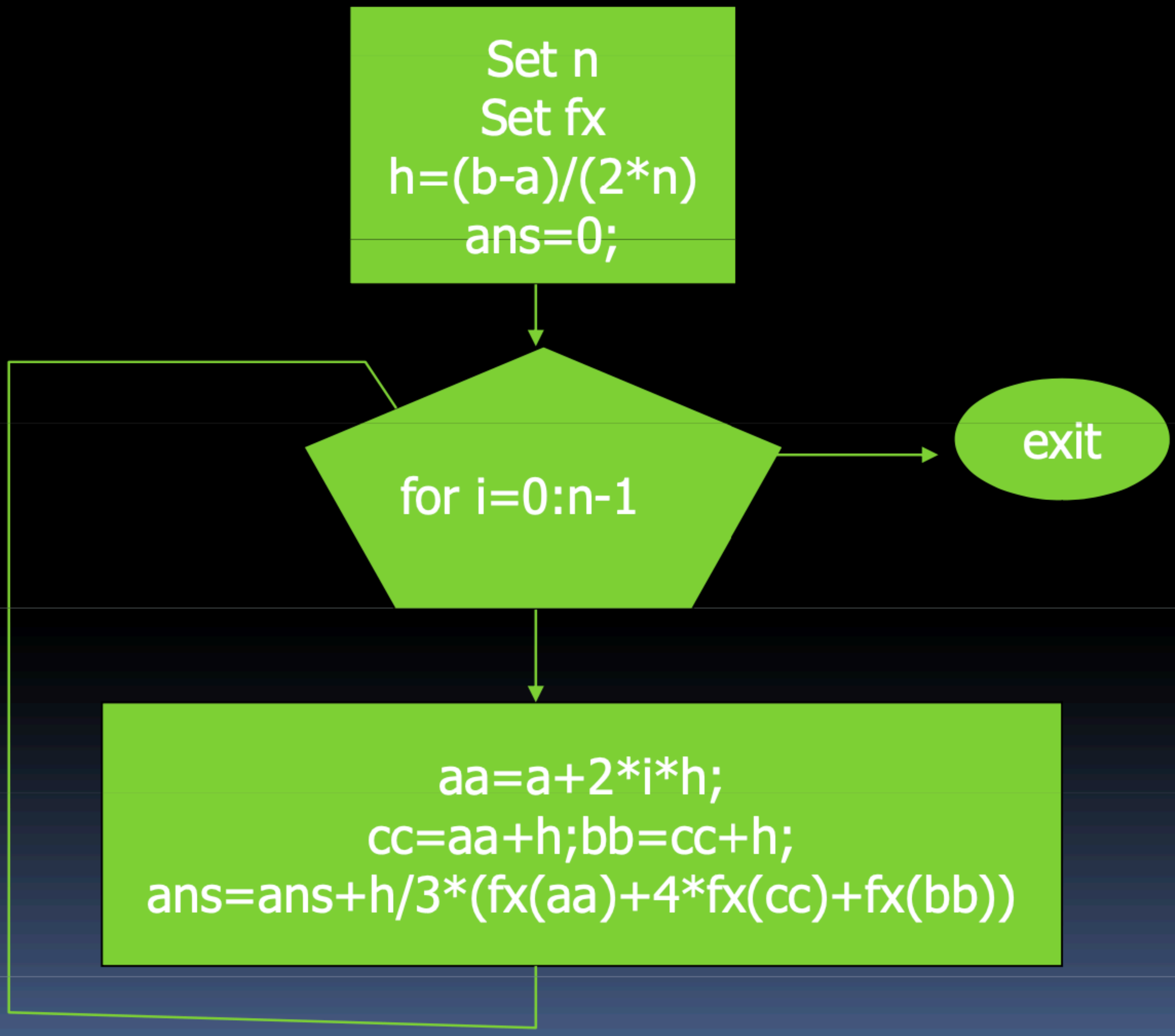
Composite Simpson rule

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{a+2ih}^{a+(2i+2)h} f(x)dx$$

$$\approx \frac{h}{3} \sum_{i=0}^{n-1} (f(a+2ih) + 4f(a+(2i+1)h) + f(a+(2i+2)h))$$

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} \int_{a+2ih}^{a+(2i+2)h} f(x)dx$$

$$\approx \frac{h}{3} \sum_{i=0}^{n-1} (f(a+2ih) + 4f(a+(2i+1)h) + f(a+(2i+2)h))$$



Composite Simpson rule

$$x_k = a + jh,\ j = 0,1,\dots,2n$$ h=(b-a)/2n

$$\int_a^b f(x)dx$$

$$\approx \frac{h}{3}\sum_{i=0}^{n-1}(f(a+2ih)+4f(a+(2i+1)h)+f(a+(2i+2)h))$$

$$= \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\dots+f(x_{2n})]$$

$$= \frac{h}{3}(f(a)+f(b))+\frac{2h}{3}\sum_{k=1}^{n-1}f(x_{2k})+\frac{4h}{3}\sum_{k=1}^{n-1}f(x_{2k-1})$$

Composite Simpson rule

$$x_k = a + jh, j = 0,1,\ldots,2n$$ h=(b-a)/2n

$$\int_a^b f(x)dx$$

$$\approx \frac{h}{3}(f(a)+f(b)) + \frac{2h}{3}\sum_{k=1}^{n-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{n-1} f(x_{2k-1})$$

Simpson's Rule for Numerical Integration