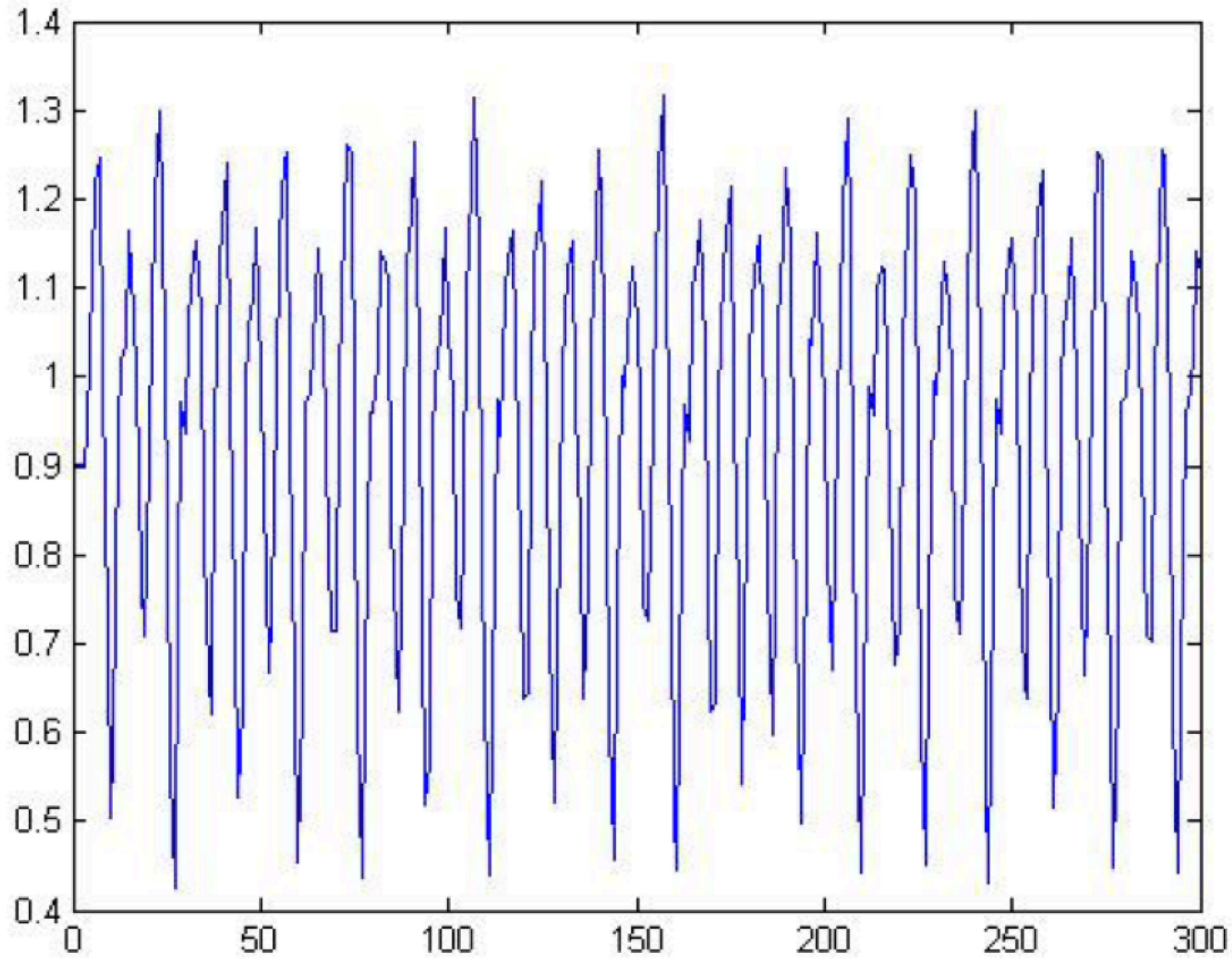
Euler Method
Runge-Kutta method for tracking differential equations

Ordinary differential equations

Chaos time series

Mackey-Glass data: 1-dimension from formula $\frac{\partial x}{\partial t} = \frac{ax(t-\tau)}{1+x^{c}(t-\tau)} - bx(t)$

x[1:300], $x = g(t)$



Analytic approach

- Good luck with previous equation, but others ...

- Analytically solvable
  - $x' - x = e^t \quad \Rightarrow \quad x(t) = t\,e^t + c\,e^t$
  - $x'' + 9x = 0 \quad \Rightarrow \quad x(t) = c_1 \sin 3t + c_2 \cos 3t$
  - $x' + \frac{1}{2x} = 0 \quad \Rightarrow \quad x(t) = \sqrt{c - t}$

First order IVP

- General form:

$$x' = f(t, x), \quad x(a) \text{ given}$$

- Note: non-linear, non-homogeneous

- Examples
  - $x' = x + 1, \quad x(0) = 0 \quad \Rightarrow \quad x(t) = e^t - 1$
  - $x' = 6t - 1, \quad x(1) = 6 \quad \Rightarrow \quad x(t) = 3t^2 - t + 4$
  - $x' = \frac{t}{x+1}, \quad x(0) = 0 \quad \Rightarrow \quad x(t) = \sqrt{t^2 + 1} - 1$

Numerical technique

- Source of need
  - ∗ Usually analytical solution is not known
  - ∗ Even if known, perhaps very complicated, expensive to compute

- Numerical techniques
  - ∗ Generate a table of values for $x(t)$
  - ∗ Usually equispaced in t, stepsize $= h$

Euler method

- First-order IVP: given $x' = f(t, x)$, $x(a)$, want $x(b)$
- Use first 2 terms of Taylor series (i.e., $n = 1$) to get from $x(a)$ to $x(a + h)$

$$x(a+h) = x(a) + h\,\underbrace{x'(a)}_{\text{use } f(a,\,x(a))} + \overbrace{O\left(h^2\right)}^{\text{truncation error}}$$

- Repeat to get from $x(a + h)$ to $x(a + 2h)$, ...
- Total $n = \frac{b-a}{h}$ steps until $x(b)$

Euler Formula

$$given \quad x(a), find \; x(b)$$

Iterative approach

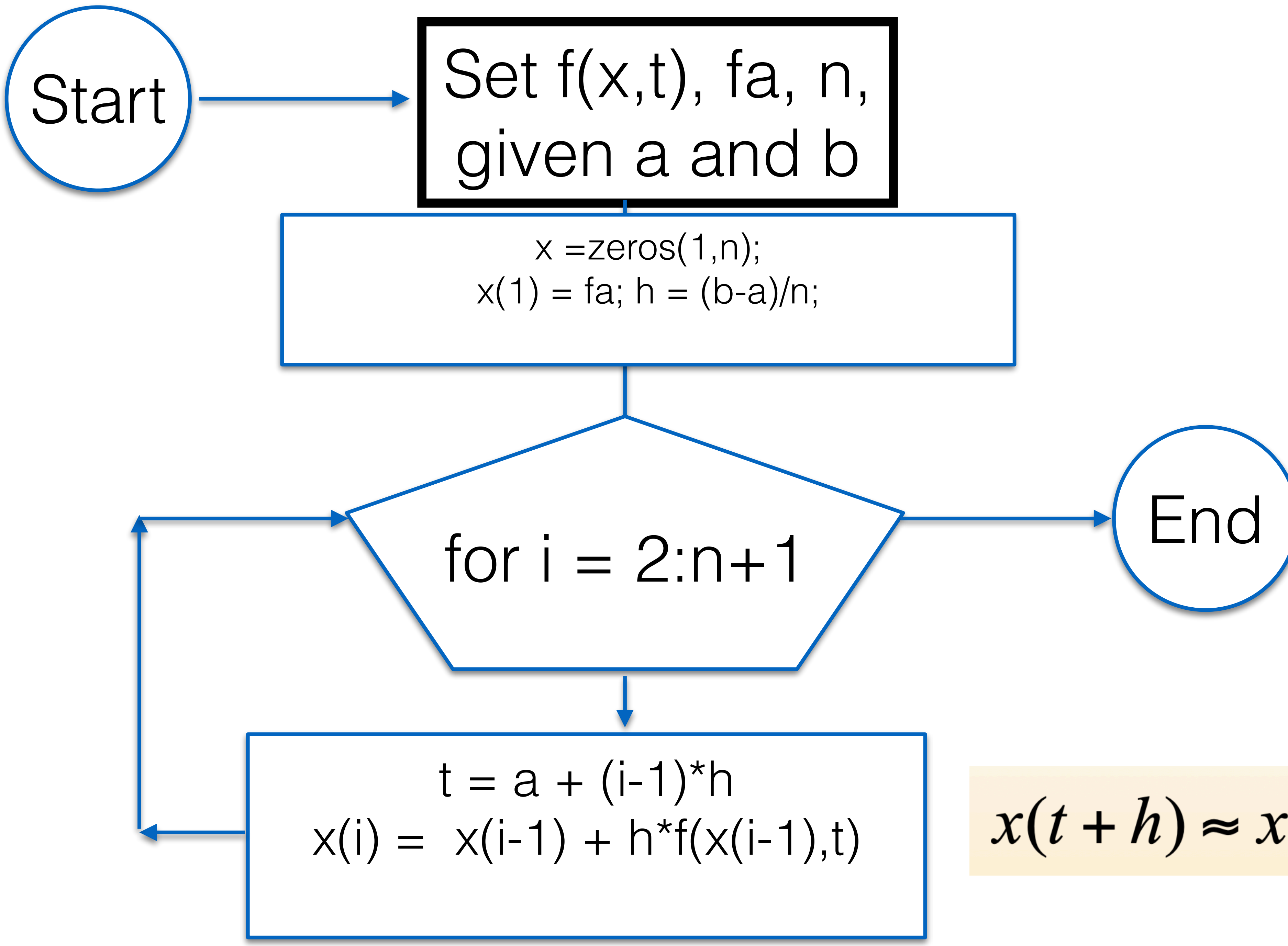
$$x(t+h) \approx x(t) + hx'(t),$$

$$t = a, a+h, a+2h, \ldots, a+(n-1)h, b$$

$$n = (b-a)/h$$

a+nh

x[1] x[2] x[n+1]



$$x(t+h) \approx x(t) + hx'(t),$$

Euler method

```
Function x=euler(fa,a,b,fx)
   n=100; h=(b-a)/n; x(1)=fa;
   for i=2:n+1
        t=a+(i-1)*h
        x(i)=x(i-1)+h*fx(x(i-1),t);
   end
```

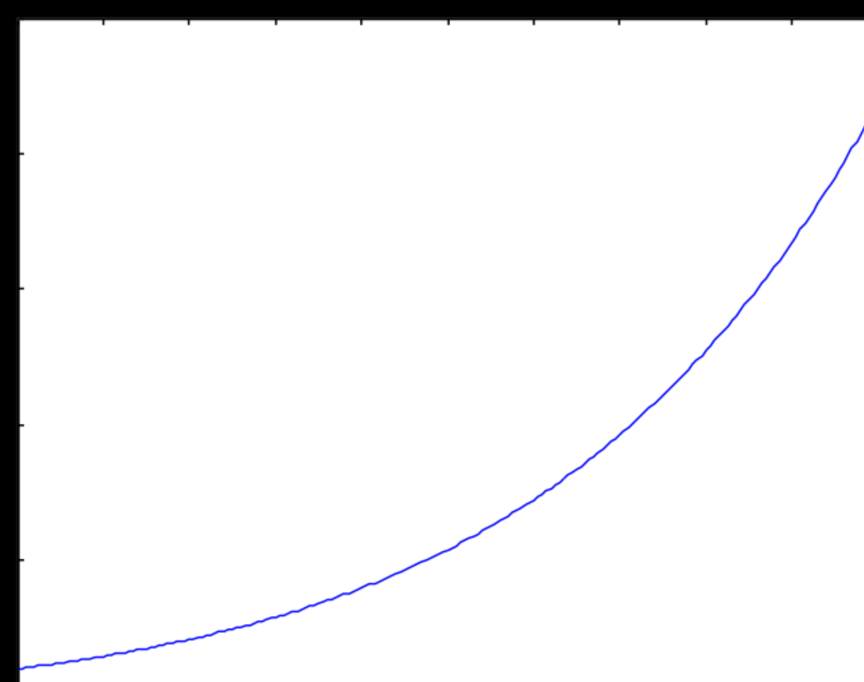
Accuracy

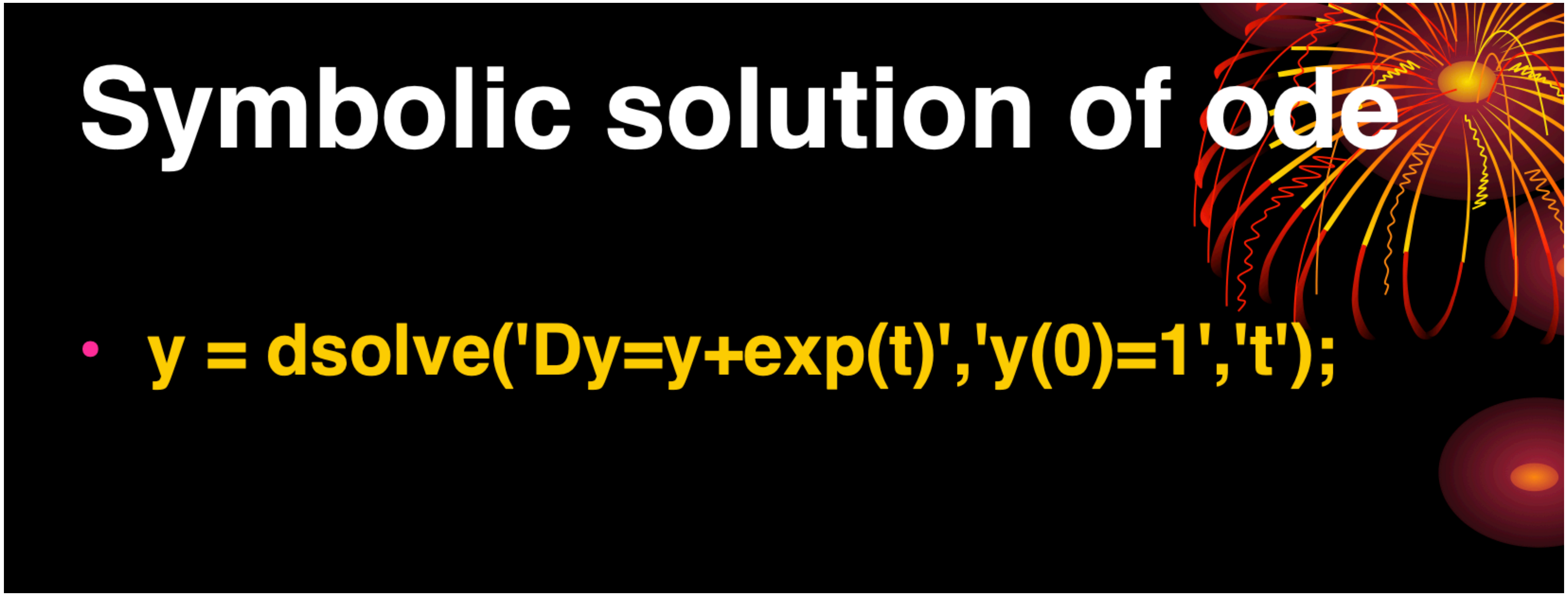
- Truncation error : $O(h^2)$
- Cumulative truncation error :
  - $O(n*h^2)$
  - $O(h)$

Example

$$* \quad x' - x = e^t \qquad \Rightarrow \qquad x(t) = t\,e^t + c\,e^t$$

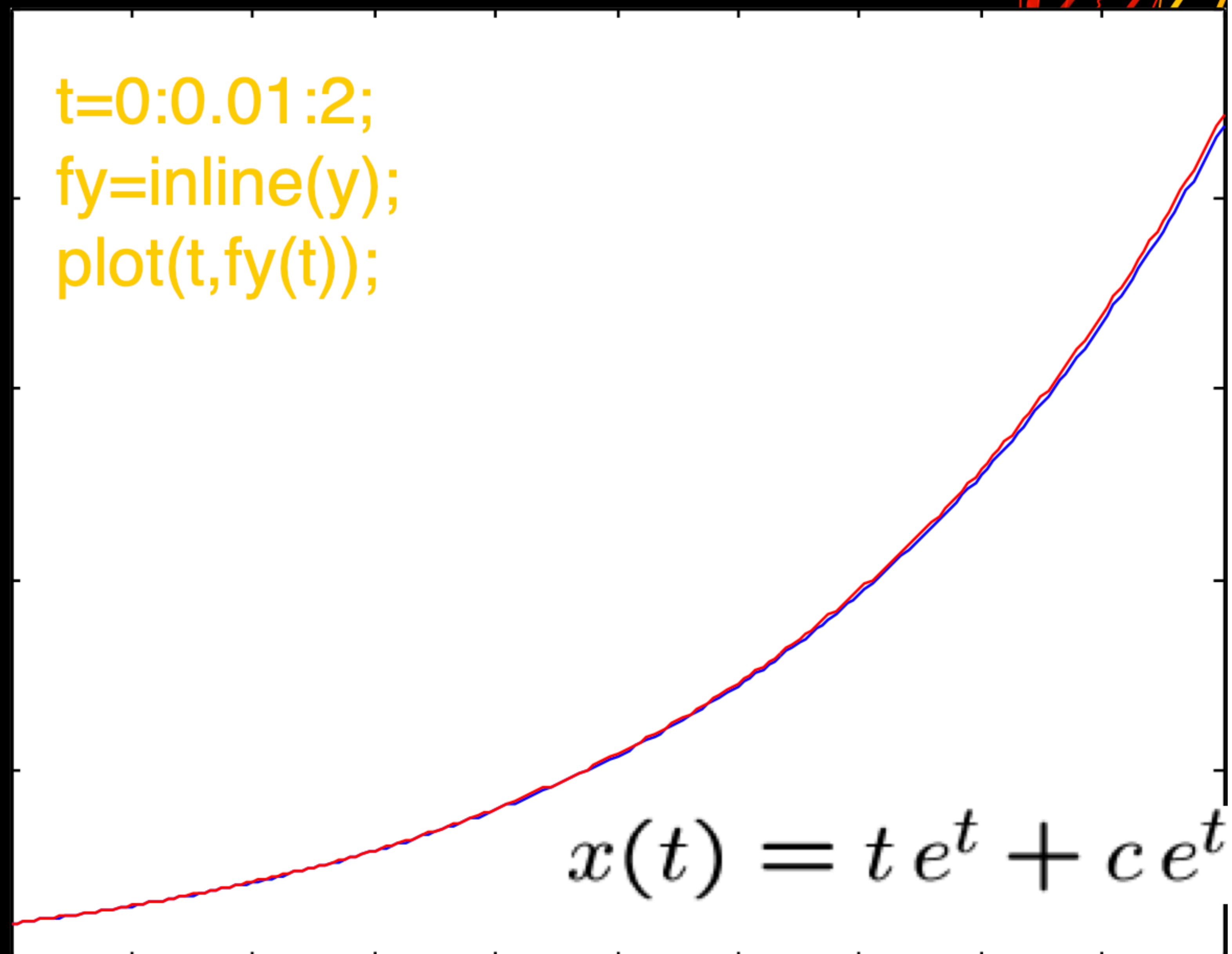
>> demo_euler
keyin number of dependent variables:2
keyin derivative function of x and t:x+exp(t)
a:0
x(a) :1
b:2
h:0.01





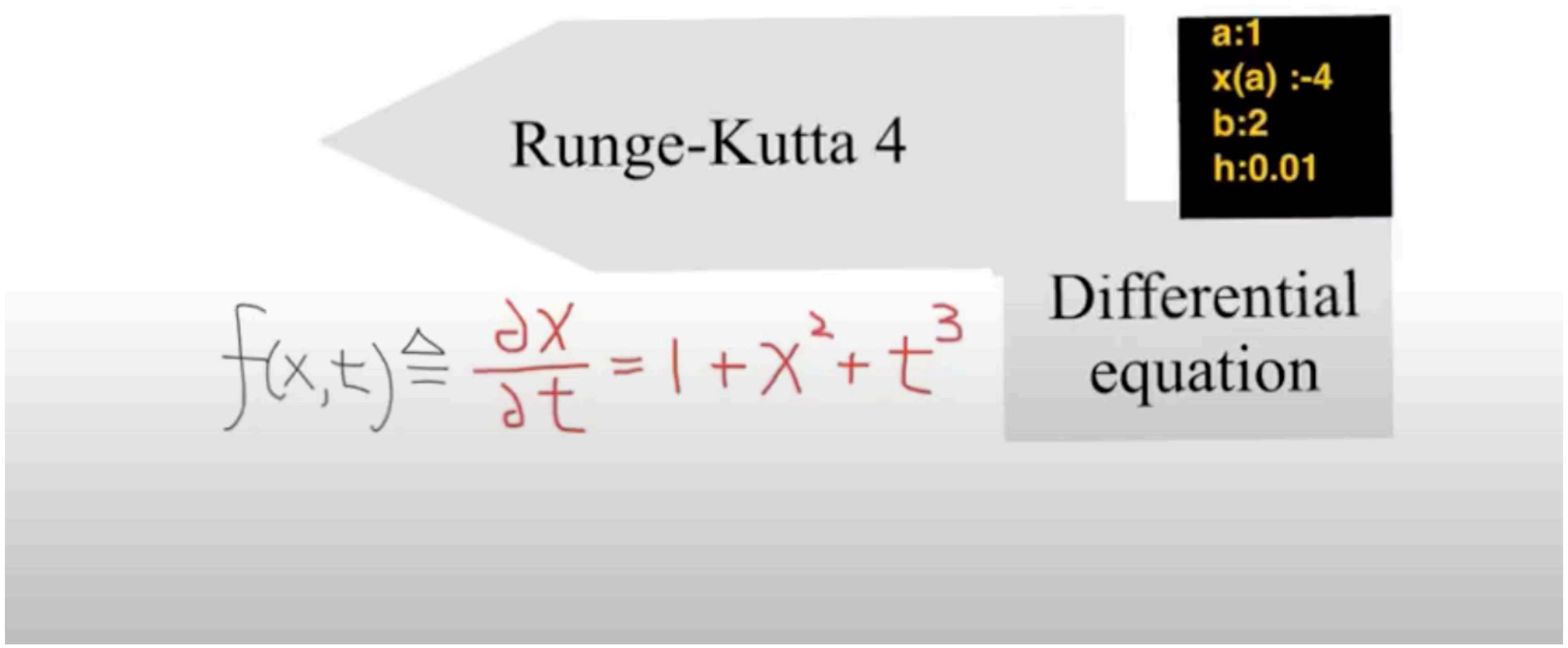
y = dsolve('Dy = y+exp(t)’,’y(0)=1','t');

Error



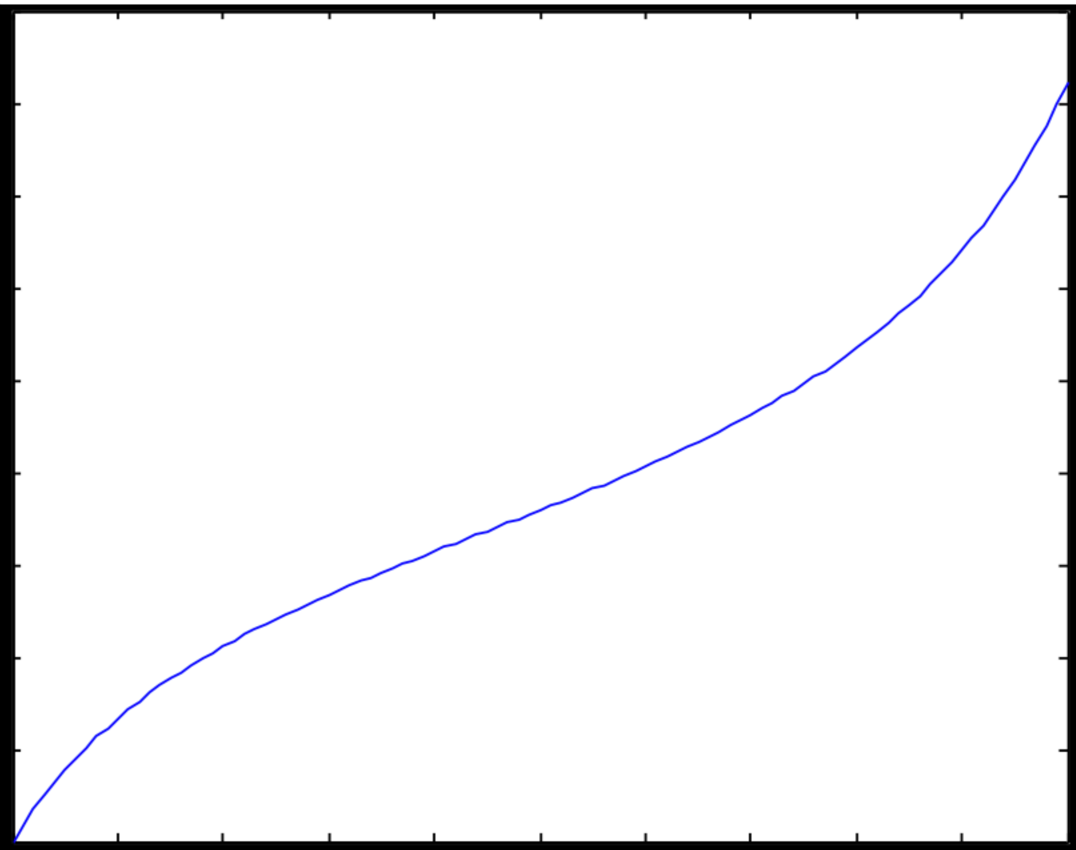
Problem 1. State the formula of the Runge-Kutta 4 method for tracking differential equations. Draw a flow chart to illustrate solving the initial value problem by Runge-Kutta4

Problem 2. Write matlab codes to implement your RK4 flow chart and test with the following IVP problem.



Example

$$x' = 1 + x^2 + t^3,\ x(1) = -4,\ \text{want } x(2)$$

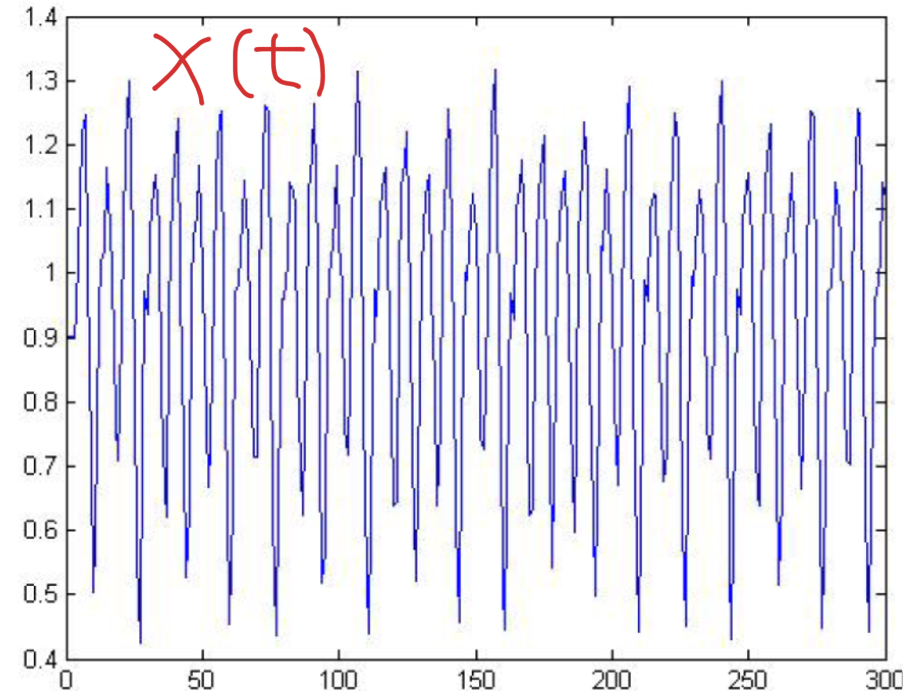


Chaotic differential equation

$$\frac{\partial x}{\partial t} = \frac{a x(t-\tau)}{1 + x^{c}(t-\tau)} - b x(t)$$

a=0.2,c=10,b=0.1, tau=17

Given a differential equation, find x(t)



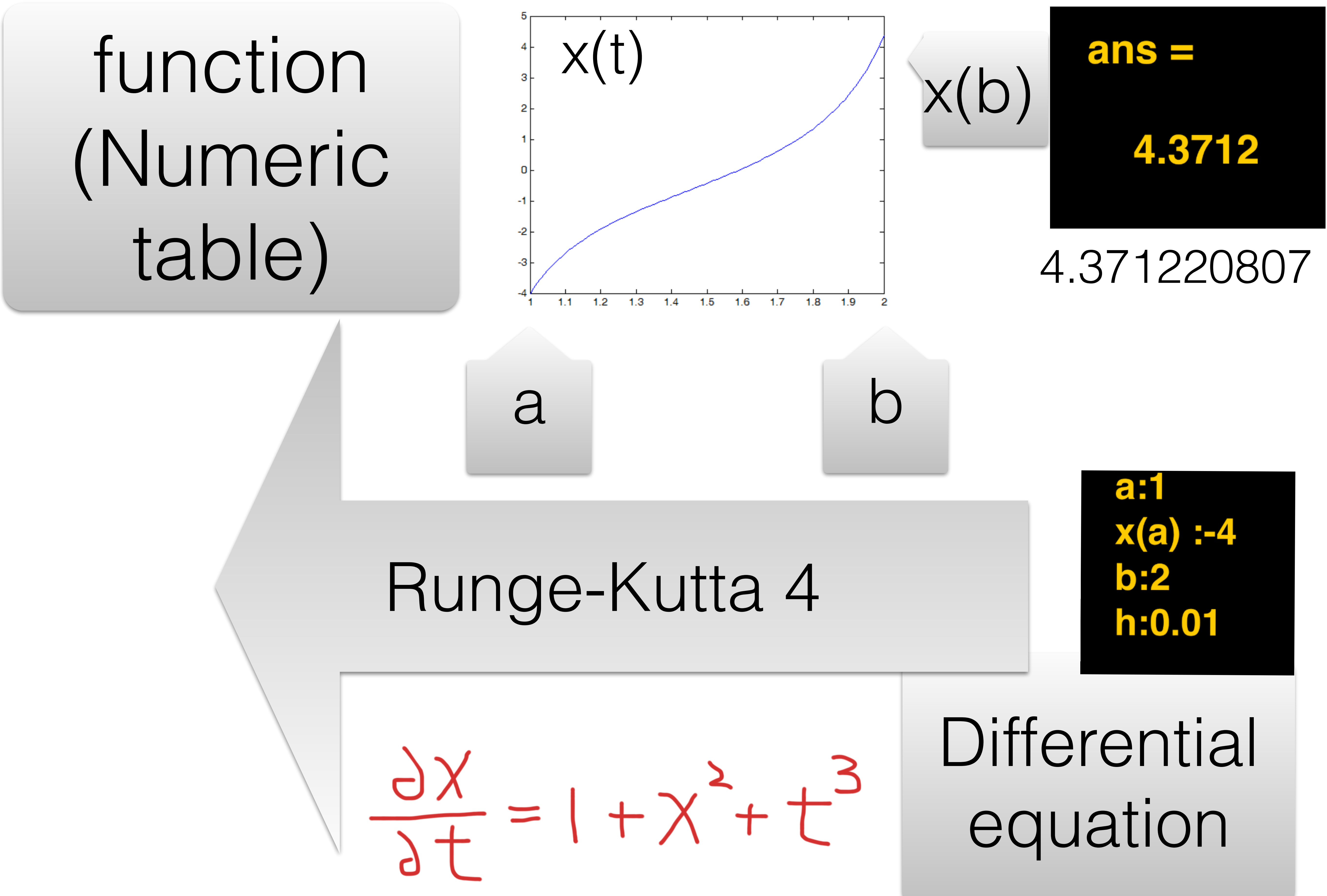
RK(Runge-Kutta)4

nonlinear delay differential equation

$$\frac{\partial x}{\partial t} = x(t - \tau) - x^3(1 - \tau)$$

where the delay τ is set to 1.6.

$$x(0) = 0.2$$



function (Numeric table)
x(t)
x(b)
ans =
4.3712
4.371220807
a
b
Runge-Kutta 4
a:1
x(a) :-4
b:2
h:0.01
$\frac{\partial x}{\partial t} = 1 + x^2 + t^3$
Differential equation

- Error for order m is $O\left(h^{m+1}\right)$ for each step of size h

2024.5.30

<u>RK4:</u>

$$x(t+h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$f(t,x) = \frac{\partial x}{\partial t} = 1 + x^2 + t^3$

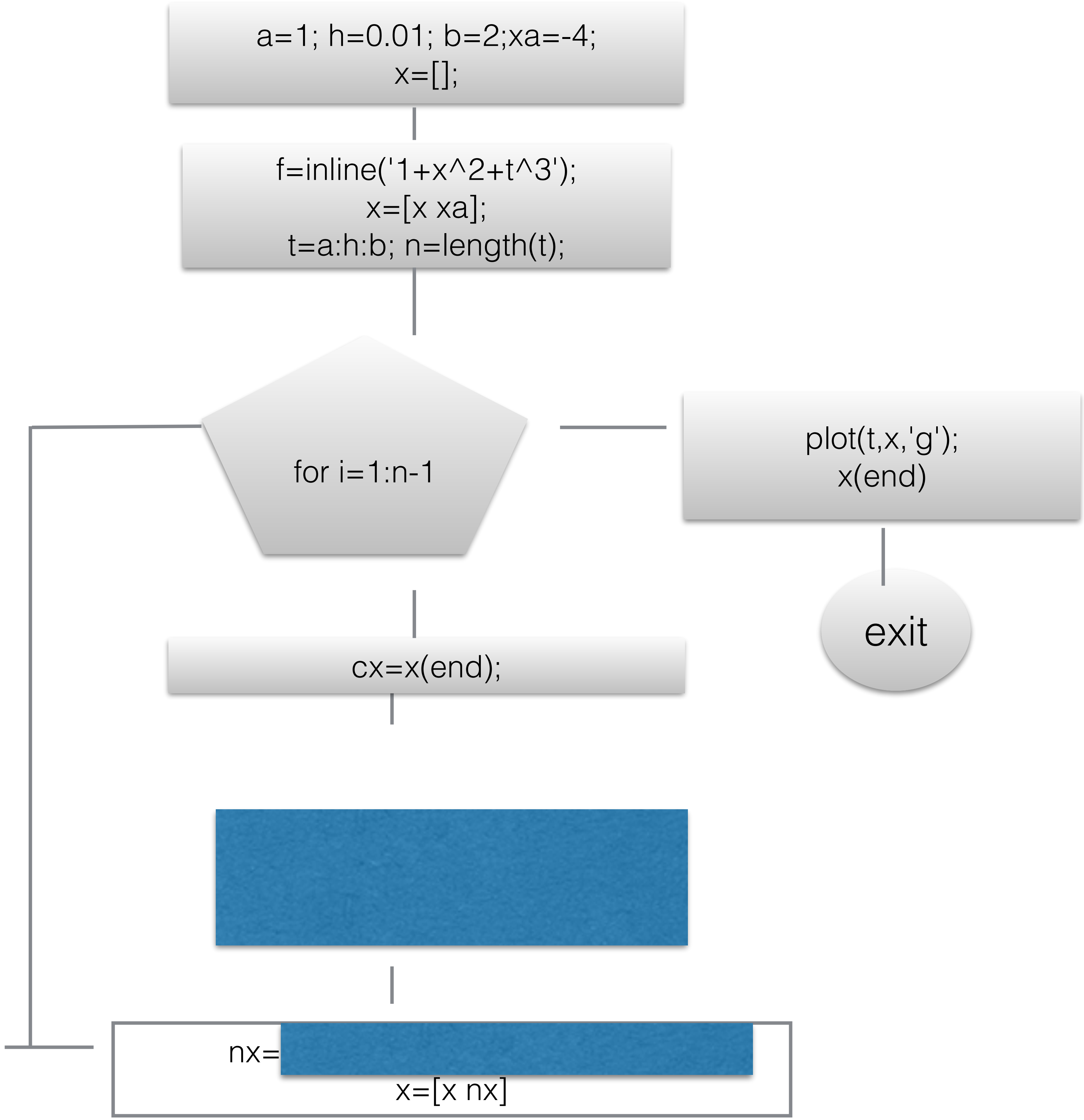
$$\begin{cases} F_1 &= hf(t,x) \\ F_2 &= hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_1\right) \\ F_3 &= hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_2\right) \\ F_4 &= hf(t+h, x+F_3) \end{cases}$$

```
a=1; h=0.01; b=2;xa=-4
x=[];

f=inline('1+x^2+t^3');
x=[x xa];
t=a:h:b; n=length(t);

for
i=1:n-1

cx = x(end)
t = a + i*h
F1 = h*f(t,x)
F2 = h*f(t+h/2,x+F1/2)
F3 = h*f(t+h/2,x+F2/2)
F4 = h * f(t+h,x+F3)
nx = cx+1/6*(F1+2*F2 +2*F3+F4)

x = [x F4]

plot(t,x,'g');
x(end)

exit
```

<u>RK4:</u>

$$x(t+h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$$\begin{cases} F_1 &= hf(t,x) \\ F_2 &= hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_1\right) \\ F_3 &= hf\left(t + \frac{1}{2}h, x + \frac{1}{2}F_2\right) \\ F_4 &= hf(t+h, x+F_3) \end{cases}$$


a=1; h=0.01; b=2;xa=-4;
x=[];
f=inline('1+x^2+t^3');
x=[x xa];
t=a:h:b; n=length(t);
for i=1:n-1
plot(t,x,'g');
x(end)
exit
cx=x(end);
nx=
x=[x nx]

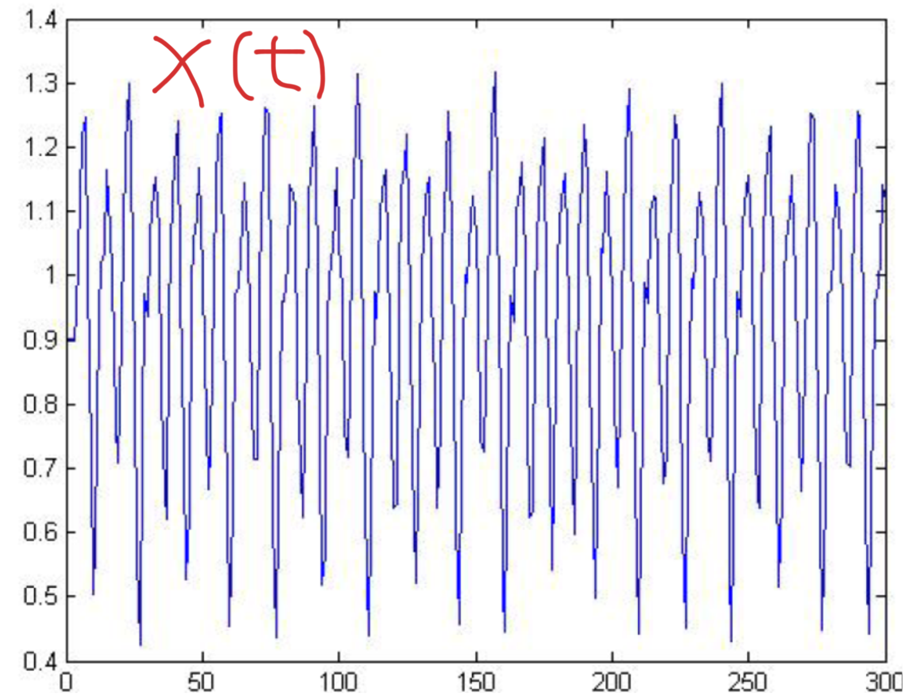
```
for i=1:n-1
cx=x(end);
F1=h*f(t(i),cx);
F2=h*f(t(i)+h/2,cx+F1/2);

nx=cx+1/6*(F1+2*F2+2*F3+F4);
x=[x nx];
end
```

Chaotic differential equation

$$\frac{\partial x}{\partial t} = \frac{a x(t-\tau)}{1 + x^c(t-\tau)} - b x(t)$$

a=0.2,c=10,b=0.1, tau=17

Given a differential equation, find x(t)



REPORT

Oscillation and Chaos in Physiological Control Systems

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Abstract

First-order nonlinear differential-delay equations describing physiological control systems are studied. The equations display a broad diversity of dynamical behavior including limit cycle oscillations, with a variety of wave forms, and apparently aperiodic or "chaotic" solutions. These results are discussed in relation to dynamical respiratory and hematopoietic diseases.

```
function demo_RK4_MG()
data=[0;0.9];  % row 1 stores t values, row 2 storing x values
h=0.1;
t=0;
for i=1:1000/h
    L=size(data,2); cx=data(2,L);
    F1=h*mg(data,t,cx);
    F2=h*mg(data,t+h/2,cx+F1/2);
    F3=h*mg(data,t+h/2,cx+F2/2);
    F4=h*mg(data,t+h,cx+F3);
    nx=cx+1/6*(F1+2*F2+2*F3+F4);
    t=t+h;
    data=[data [t nx]'];
end
plot(data(1,:),data(2,:),'b');
return

function ans=mg(data,t,x)
if t-17 < 0
    px = data(2,1);
else
    d=abs(data(1,:)-(t-17));
    [v ind]=min(d);
    px=data(2,ind);
end
ans=0.2*px/(1+px^10)-0.1*x;
```

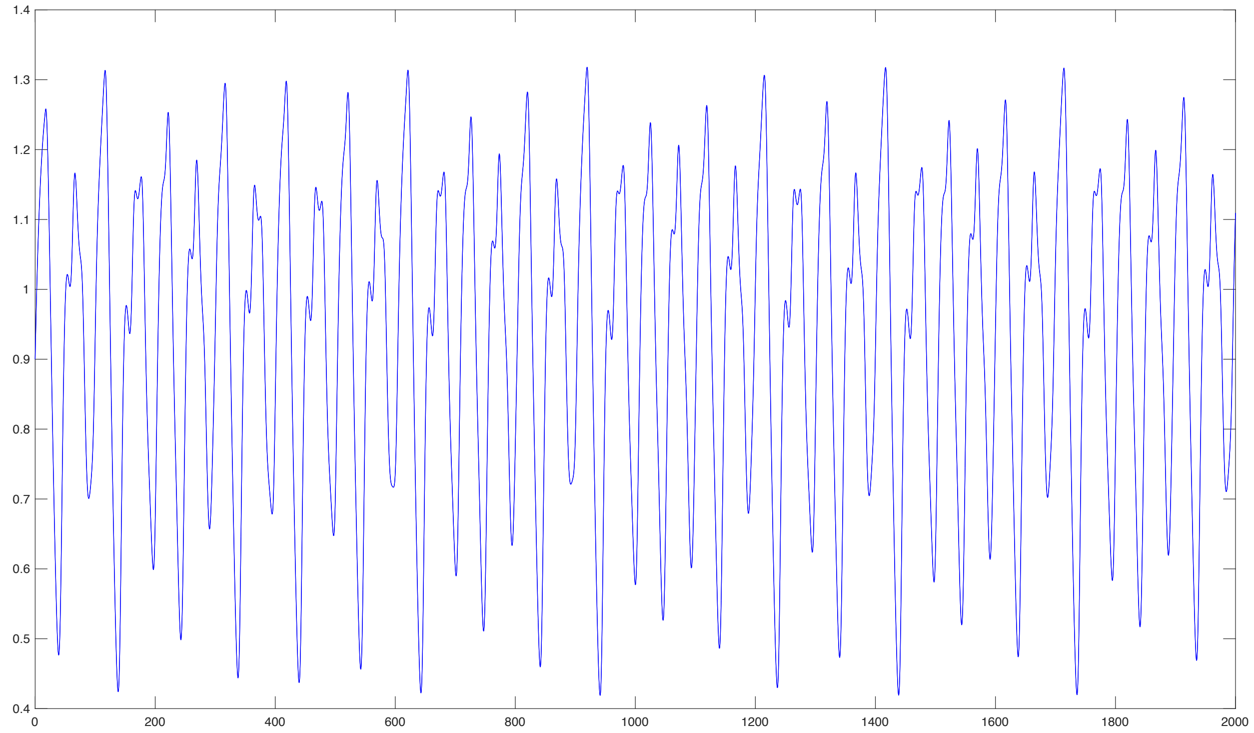
$$\frac{\partial x}{\partial t} = \frac{ax(t-\tau)}{1+x^c(t-\tau)} - bx(t)$$

a=0.2,c=10,b=0.1, tau=17

<u>RK4:</u>

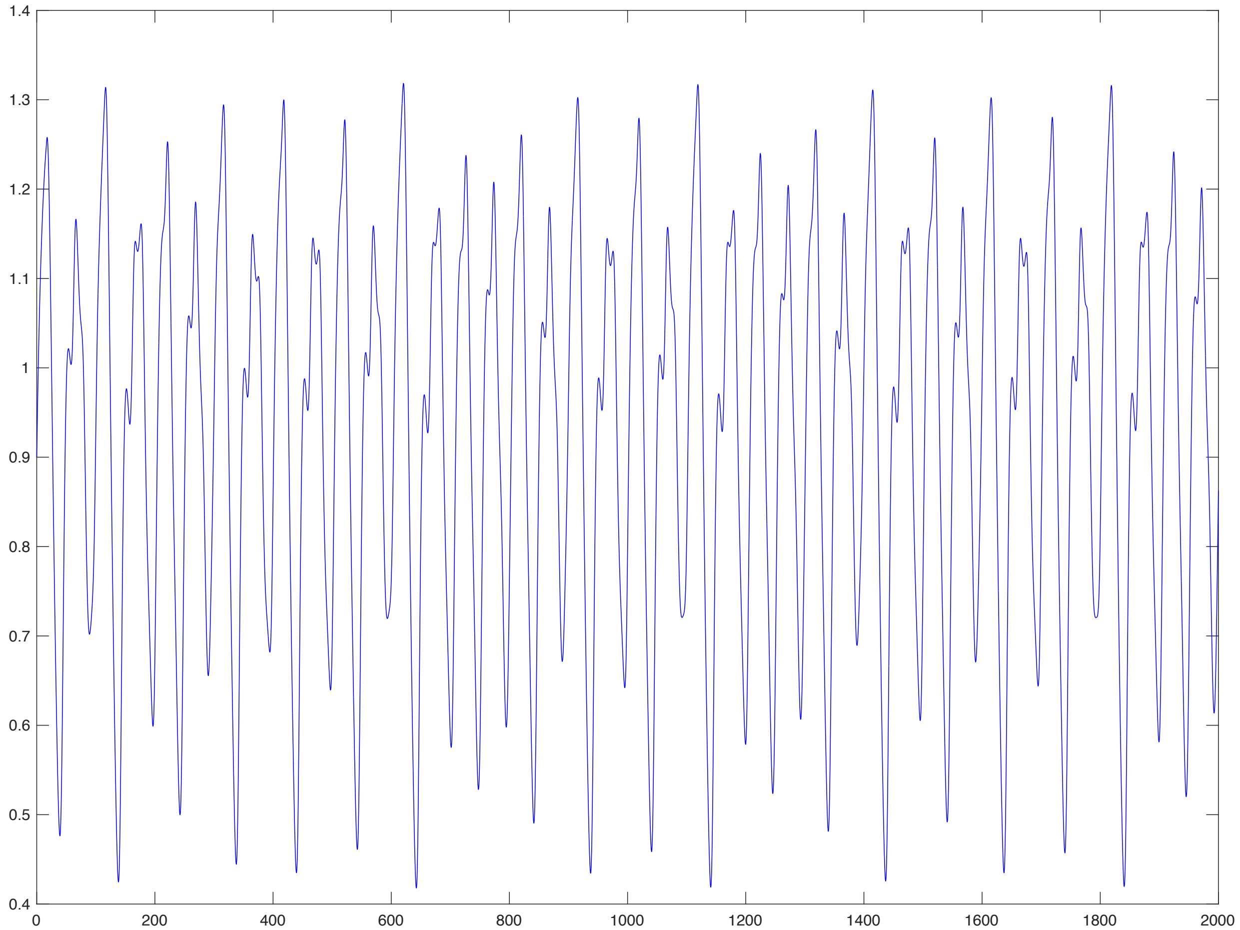
$$x(t+h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$$\begin{cases} F_1 &= hf(t,x) \\ F_2 &= hf\left(t+\frac{1}{2}h, x+\frac{1}{2}F_1\right) \\ F_3 &= hf\left(t+\frac{1}{2}h, x+\frac{1}{2}F_2\right) \\ F_4 &= hf(t+h, x+F_3) \end{cases}$$



h = 0.1

1.10931531



h = 0.05

0.86264378

RK(Runge-Kutta)4

nonlinear delay differential equation

$$\frac{\partial x}{\partial t} = x(t-\tau) - x^3(1-\tau),$$

where the delay τ is set to 1.6.

$$x(0) = 0.2$$