# Linear system solving

An algorithm to find Reduced Echelon Form

# Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

**Solution**

pivot

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} \textcircled{3} & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \xrightarrow{(1/3)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

$$\xrightarrow{R3+(-4)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & \textcircled{2} & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow{(1/2)R2} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

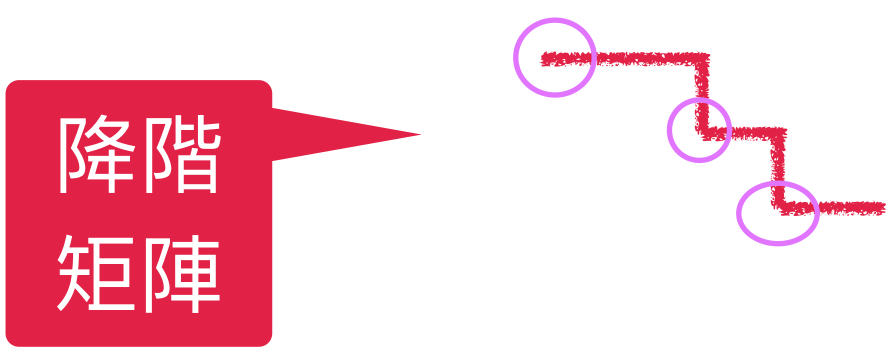
pivot

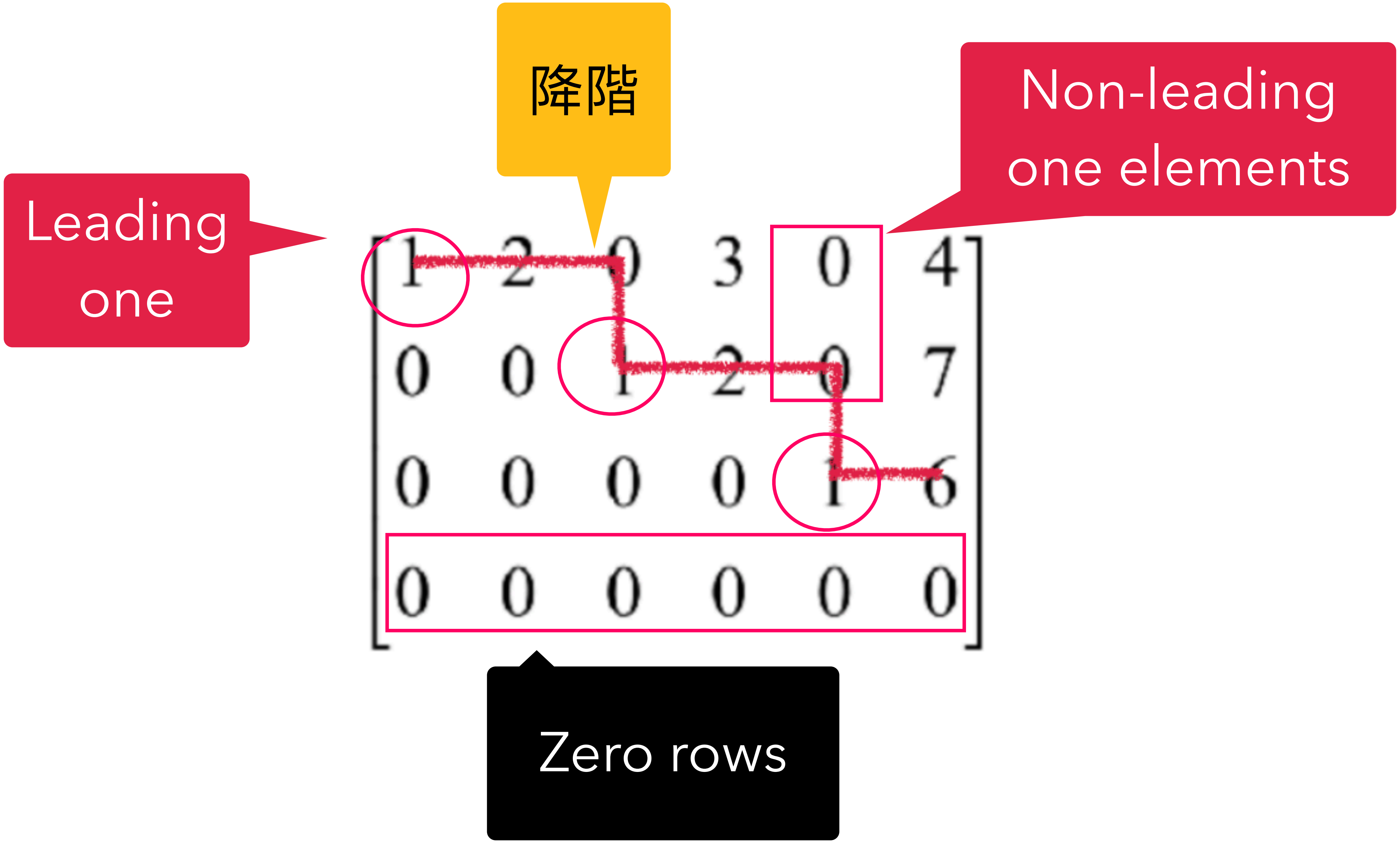
$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow[R3+(-2)R2]{R1+R2} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

pivot

$$\xrightarrow[R2+R3]{R1+(-2)R3} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

The matrix is the reduced echelon form of the given matrix.




降階
Leading one
Non-leading one elements
1 2 0 3 0 4
0 0 1 2 0 7
0 0 0 0 1 6
0 0 0 0 0 0
Zero rows

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12
```

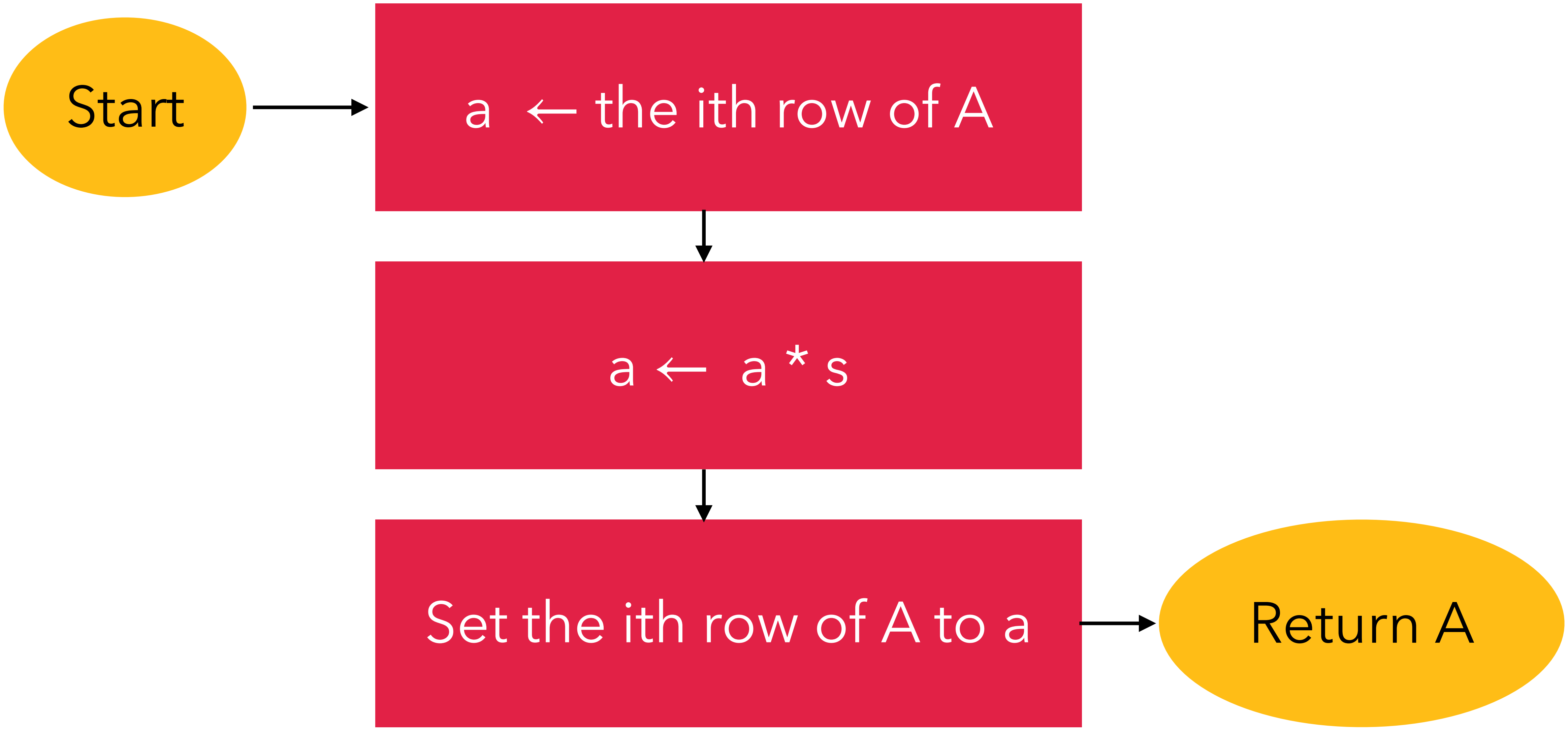
# Row operation I

Multiply elements of a row by a non-zero constant

```
function  A = row_op1(A, i, s)
% multiply s to row i
```

Exercise 1
Write a Matlab function for row_op1

```
function  A = row_op1(A, i, s)
% multiply s to row i
```



```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12

>> A = row_op1(A, 2, 2)

A =

     0     0     2    -2     2
     6     6    -6    18    24
     4     4    -2    11    12
```

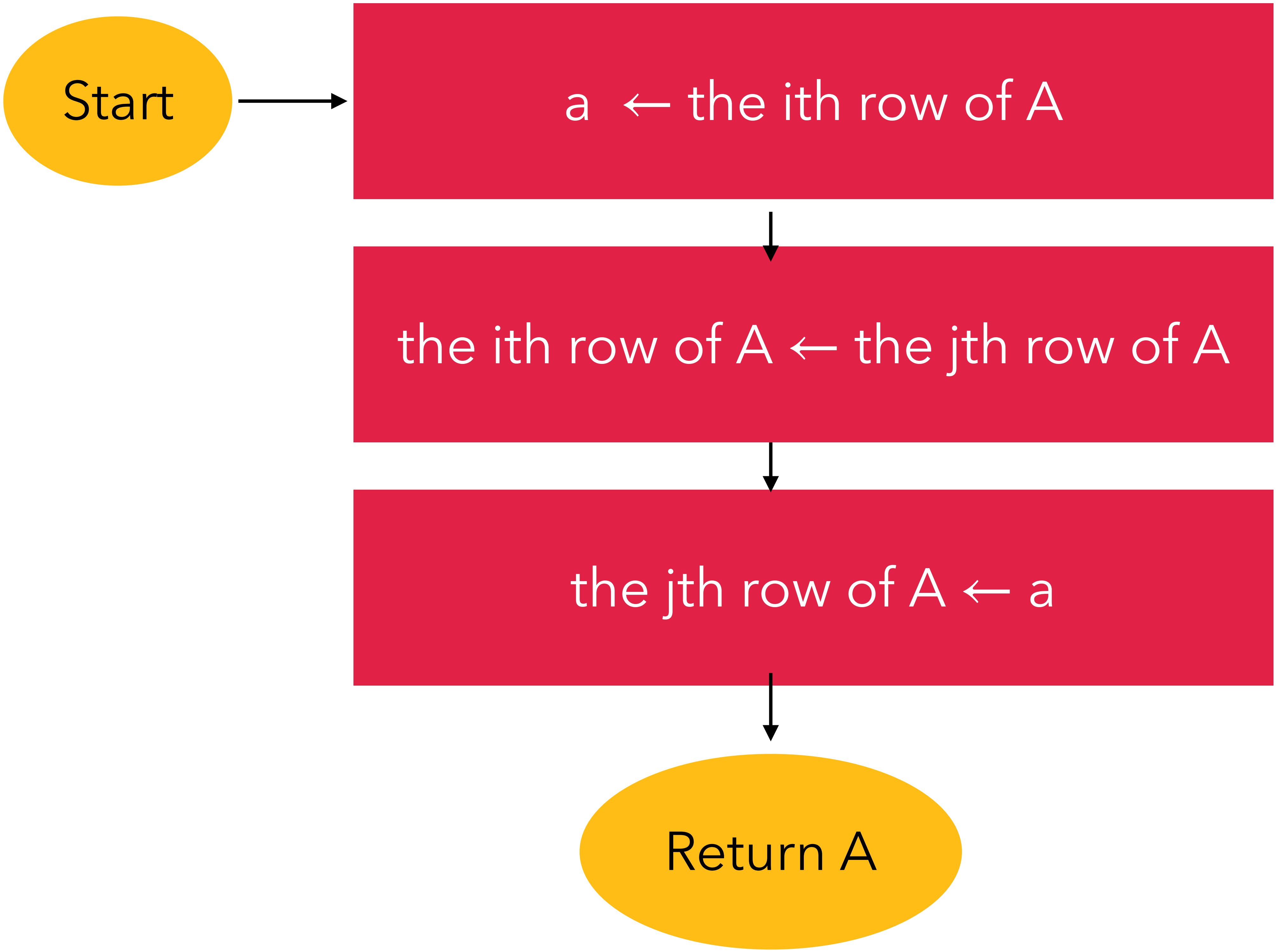
Exercise 2
Give an example to test row_op1

# Row operation II
# Swap row i and row j

```
function  A = row_op2(A, i, j)
% swap row i and row j
```

Exercise 3
Write a Matlab function for row_op2

```
function A = row_op2(A, i, j)
% swap row i and row j
```



```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12
```

Exercise 4
Give an example to test row_op2

```
>> A = row_op2(A,1,2)

A =

     3     3    -3     9    12
     0     0     2    -2     2
     4     4    -2    11    12
```

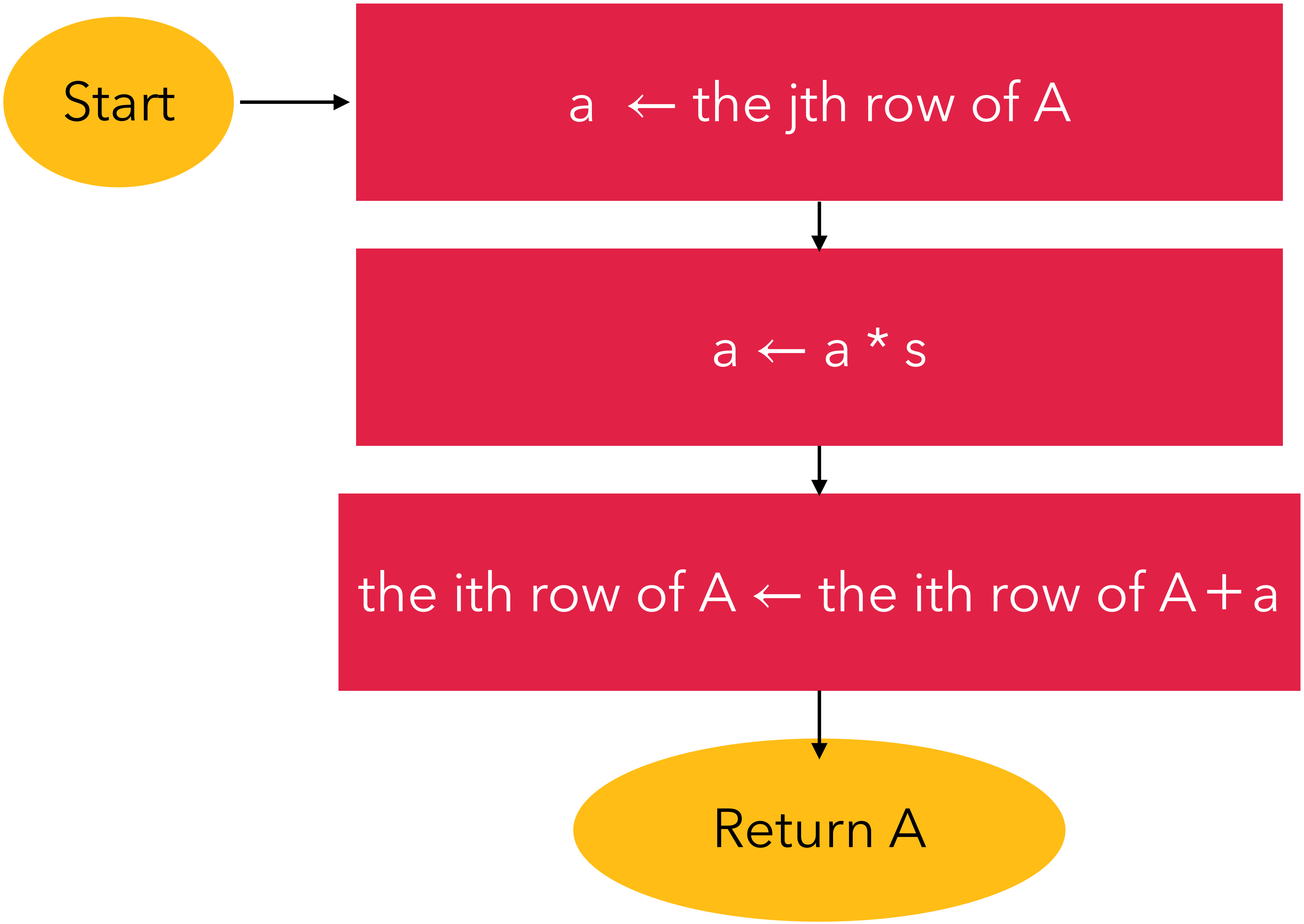
# Row operation III

Add a multiple of row j to row i

```
function  A = row_op3(A, i, j, s)
% add a multiple of row j to
% row i
```

Exercise 5
Write a Matlab function for row_op3

```
function  A = row_op3(A, i, j, s)
% add a multiple of row j to
% row i
```



```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12

>> A = row_op3(A,2, 3, 2)

A =

     0     0     2    -2     2
    11    11    -7    31    36
     4     4    -2    11    12
```

Exercise 6
Give an
example to
test row_op3

# Gauss-Jordan Elimination

- To solve a system of equations, we can perform **elementary row operations**
  - Interchange two rows of a matrix
  - Multiply the elements of a row by a nonzero constant
  - Add a multiple of the elements of one row to the corresponding elements of another row

# Example 1

Use the method of Gauss-Jordan elimination to find reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

**A = row_op2(A,1,2);**
**A= row_op1(A,1,1/3);**
**A=row_op3(A, 3,1, -4);**
**A=row_op1(A,2,1/2)**

**Solution**

**A = row_op2(A,1,2);**

pivot

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} \textcircled{3} & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix} \xrightarrow{(1/3)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

**A= row_op1(A,1,1/3);**

$$\xrightarrow{R3+(-4)R1} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & \textcircled{2} & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow{(1/2)R2} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

**A=row_op3(A, 3,1, -4);**

pivot

**A=row_op1(A,2,1/2)**

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =

     0     0     2    -2     2
     3     3    -3     9    12
     4     4    -2    11    12

>> A = row_op2(A,1,2);
A= row_op1(A,1,1/3);
A=row_op3(A, 3,1, -4);
A=row_op1(A,2,1/2)

A =

     1     1    -1     3     4
     0     0     1    -1     1
     0     0     2    -1    -4
```

Exercise 7
Give row operations to find rref of matrix A

```
function A = myFunction()
    A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12];
    A = rowop2(A,2,3);
end
function A = rowop2(A,i,j)
   a = A(i,:);
   A(i,:) = A(j,:);
   A(j,:) = a;

end
```

A = row_op3(A,1,2,1);
A=row_op3(A,3,2,-2);
A=row_op3(A,1,3,-2);
A=row_op3(A,2,3,1)

A = row_op3(A,1,2,1);

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \xrightarrow[R3+(-2)R2]{R1+R2} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

pivot

A=row_op3(A,3,2,-2);

A=row_op3(A,1,3,-2);

$$\xrightarrow[R2+R3]{R1+(-2)R3} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

A=row_op3(A,2,3,1)

The matrix is the reduced echelon form of the given matrix.

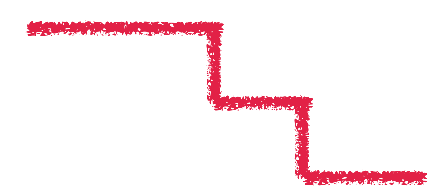
```
>> A = row_op3(A,1,2,1);
A=row_op3(A,3,2,-2);
A=row_op3(A,1,3,-2);
A=row_op3(A,2,3,1)

A =

     1     1     0     0    17
     0     0     1     0    -5
     0     0     0     1    -6
```

Exercise 7 Give row operations to find rref of matrix A

## Definition

A matrix is in **reduced echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is 1. This element is called a **leading 1**.
3. The leading 1 of each after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.



A. Zero rows at the bottom of a matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

A =

$$\begin{matrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{matrix}$$

B. Each non-zero row has a leading one

**A =**

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 17 |
| 0 | 0 | 1 | 0 | -5 |
| 0 | 0 | 0 | 1 | -6 |

C. The leading one of row i is positioned to the right of the leading one of row i-1 for i > 1

A =

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 17 |
| 0 | 0 | 1 | 0 | -5 |
| 0 | 0 | 0 | 1 | -6 |

D. If a column contains a leading one, its elements that are not the leading one must be zero

# Sub problem 1

- Check the first condition. Are any rows consisting entirely of zeros grouped at the bottom of the matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

not in ref

# Sub problem 2

- Check condition 2. Is the first nonzero element of each other row a leading one ?

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First nonzero element in row 2 is not 1

not in ref

# Sub problem 3

## Is the leading 1 of each after the first positioned to the right of the leading 1 of the previous row?

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Leading 1 in row 3 not to the right of leading 1 in row 2

not in ref

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Sub problem 4

**Check if all other elements in a column, which contains a leading 1, are zero.**

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nonzero element above leading 1 in Row 2

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

not in ref

# In Reduced Echelon Form

**A**

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

**B**

$$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

**C**

$$\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**D**

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**E**

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

**F**

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Not in Reduced Echelon Form

**G**

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row of zeros not at bottom of matrix

**H**

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First nonzero element in row 2 is not 1

**I**

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Leading 1 in row 3 not to the right of leading 1 in row 2

**J**

$$\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nonzero element above leading 1 in Row 2

```
function B=my_rref(A)
```

```
B=A;
[M N]=size(B);
pivot_col = 0;
```

```
for i = 1:M
    for j=pivot_col+1:N

        find the first non-zero element of col j from row i to row M

        if such element exist

            use it to create a leading 1 for row i

            eliminate other elements in column j

            pivot_col=j;

            break

        end

    end
end
```

**Exercise 8**
Write a Matlab function to implement my_rref

find the first non-zero element of col j from row i to row M

```
>> A = [ 0 0 2 -2 2;3 3 -3 9 12;4 4 -2 11 12]

A =        j:2

        0     0     2    -2     2
i:2 →   3     3    -3     9    12
        4     4    -2    11    12

>> i = 2; M = 3; j = 2;
>> ind = find(A(i:M,j)~=0);
>> length(ind)

ans =

     2
```

i: 2
M: 3
j: 2
Such element exists

find the first non-zero element of col j from row i to row M

```
>> A = [ 1 2 0 0 1;0 0 0 1 1; 0 0 0 1 0; 0 0 1 1 1]

         j:2
A =

     1     2     0     0     1
i:2  0     0     0     1     1
     0     0     0     1     0
     0     0     1     1     1

>> ind = find(A(2:4,2)~=0);
>> length(ind)

ans =

     0
```

i: 2
M: 4
j: 2
Such element does not exist

find the first non-zero element of col j from row i to row M

```
>> A = [ 1 2 0 0 1;0 0 0 1 1; 0 0 0 1 0; 0 0 1 1 1]
```

j:3

```
A =

     1     2     0     0     1
     0     0     0     1     1
     0     0     0     1     0
     0     0     1     1     1
```

i:2

```
>> i = 2; M = 4; j = 3;
>> ind = find(A(i:M,j)~=0);
>> length(ind)

ans =

     1
```

Such element exists

```
>> A(i-1+ind(1),j)

ans =

     1
```

The first non-zero element

**How to use such element to create a leading in row i?**

Apply row_op2 to swap row i-1+ind(1) and row i
Apply row_op1 to create a leading 1

```
>> B = my_rref(A)

B =

     1     1     0     0    17
     0     0     1     0    -5
     0     0     0     1    -6
```

Exercise 9 Give two examples to test my_rref

Determine the inverse of the matrix

$$A=\begin{bmatrix}1 & -1 & -2\\ 2 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$$

**Solution**

$$[A:I_n]=\begin{bmatrix}1 & -1 & -2 & 1 & 0 & 0\\ 2 & -3 & -5 & 0 & 1 & 0\\ -1 & 3 & 5 & 0 & 0 & 1\end{bmatrix}$$

$$\underset{\substack{R2+(-2)R1\\ R3+R1}}{\approx}\begin{bmatrix}1 & -1 & -2 & 1 & 0 & 0\\ 0 & -1 & -1 & -2 & 1 & 0\\ 0 & 2 & 3 & 1 & 0 & 1\end{bmatrix}$$

$$\underset{(-1)R2}{\approx}\begin{bmatrix}1 & -1 & -2 & 1 & 0 & 0\\ 0 & 1 & 1 & 2 & -1 & 0\\ 0 & 2 & 3 & 1 & 0 & 1\end{bmatrix}$$

```
>> A=[1 -1 -2;2 -3 -5;-1 3 5]; my_rref([A eye(3)])

ans =

     1     0     0     0     1     1
     0     1     0     5    -3    -1
     0     0     1    -3     2     1

>> inv(A)

ans =

         0    1.0000    1.0000
    5.0000   -3.0000   -1.0000
   -3.0000    2.0000    1.0000
```

Exercise 10
Find inervse of A
using my_rref

# Draw a flow chart to check type I condition of rref, and write matlab to implement the flow chart

# Sub problem 1

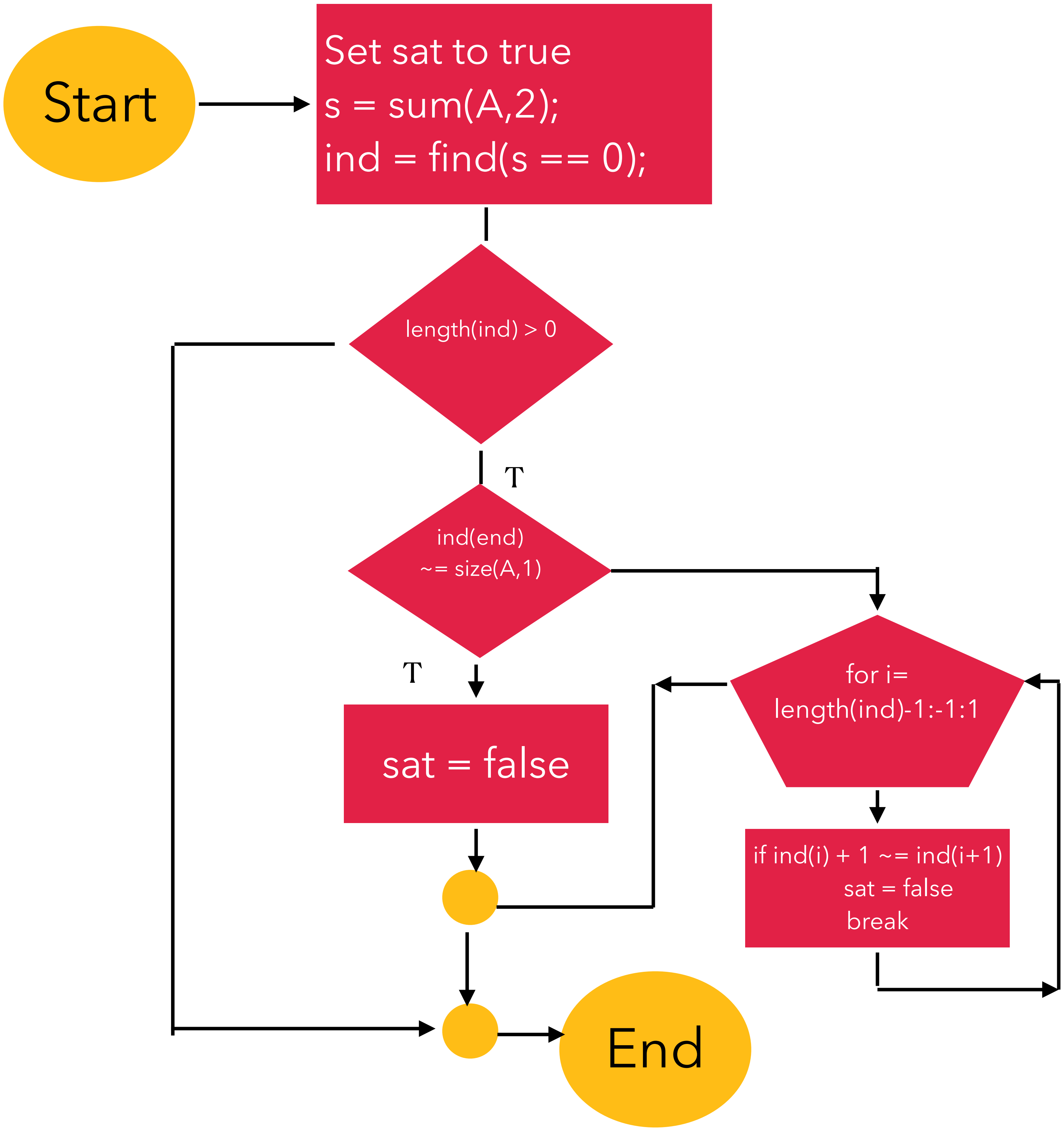
- Check the first condition. Are any rows consisting entirely of zeros grouped at the bottom of the matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

not
in ref

# function sat = check_cond1(A)

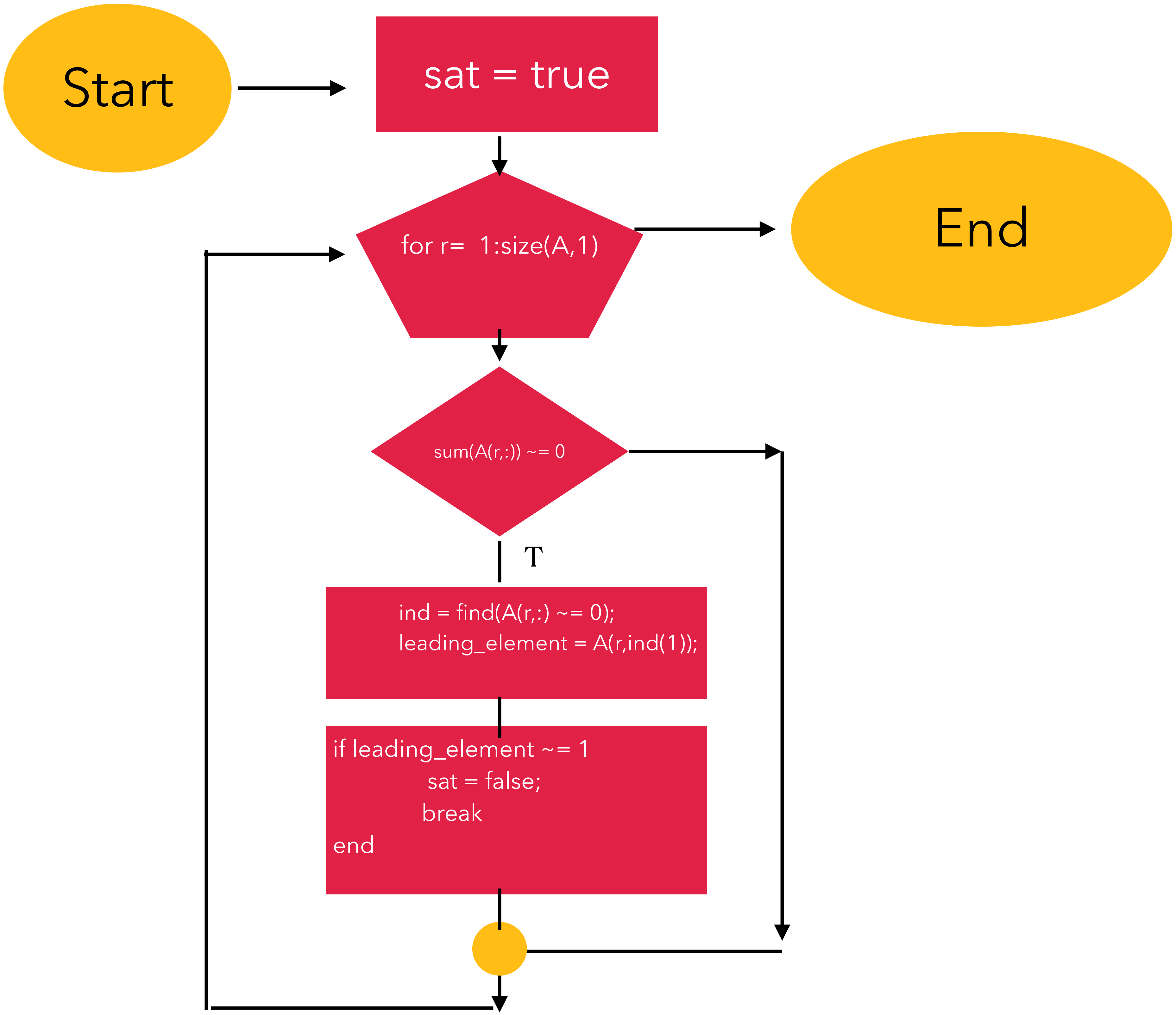
1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.



# Draw a flow chart to check type II condition of rref, and write matlab to implement the flow chart

# function sat = check_cond2(A)

2. The first nonzero element of each other row is 1. This element is called a **leading 1**.



# Sub problem 2

- Check condition 2. Is the first nonzero element of each other row a leading one ?

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

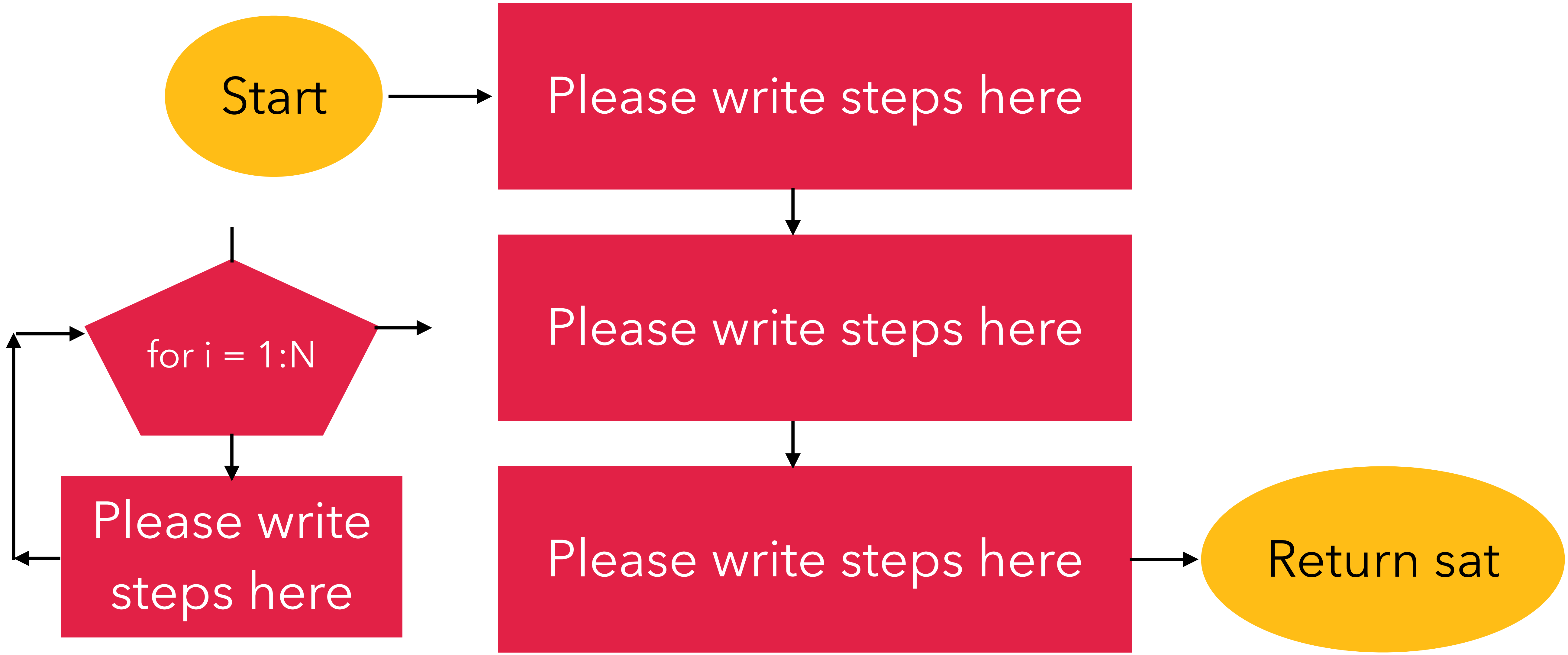
First nonzero element in row 2 is not 1

not in ref

# Draw a flow chart to check type III condition of rref, and write matlab to implement the flow chart

# function sat = check_cond3(A)

3. The leading 1 of each after the first is positioned to the right of the leading 1 of the previous row.



# Sub problem 3

**Is the leading 1 of each after the first positioned to the right of the leading 1 of the previous row?**

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Leading 1 in row 3 not to the right of leading 1 in row 2

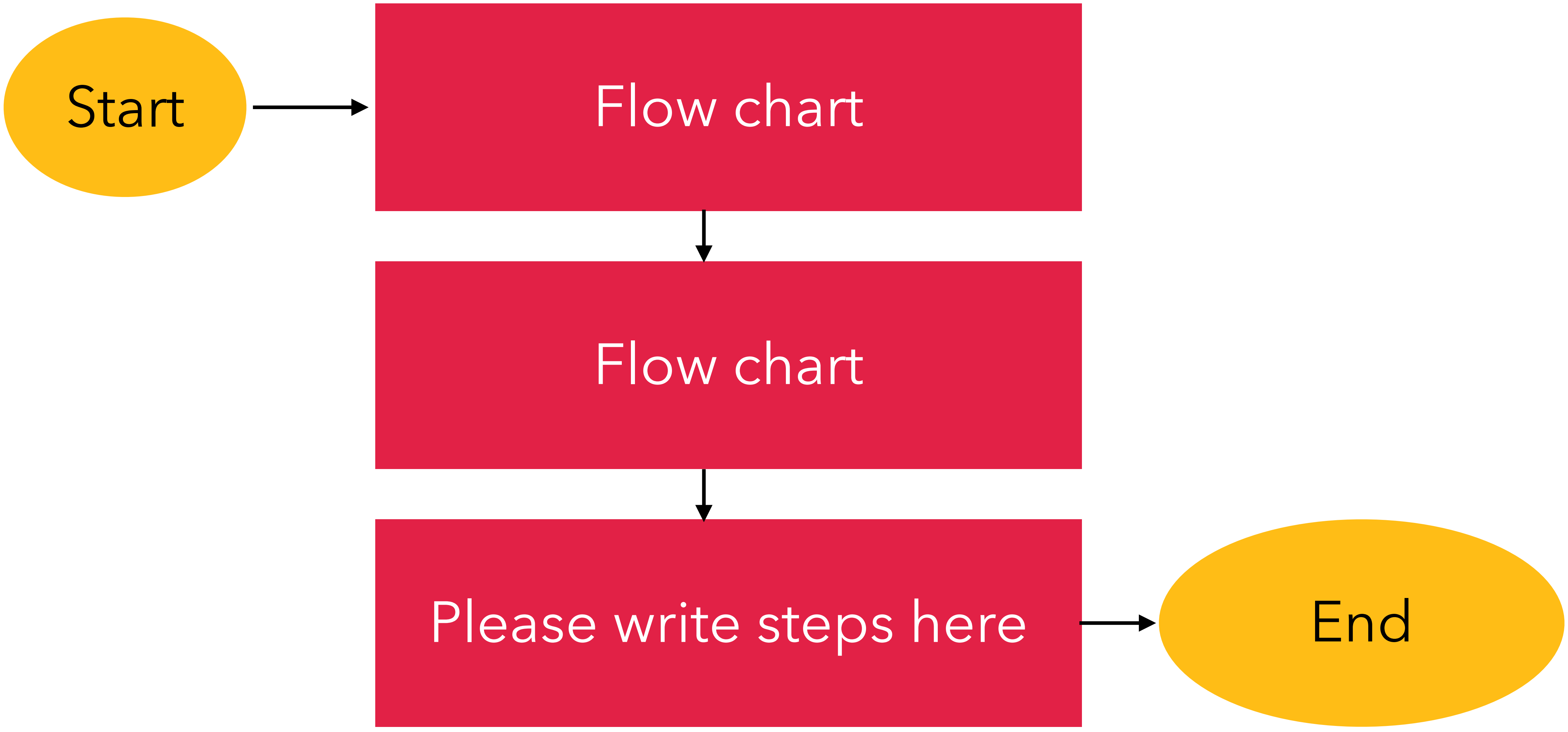
not in ref

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

# Draw a flow chart to check type IV condition of rref, and write matlab to implement the flow chart

function sat = check_cond4(A)

4. All other elements in a column that contains a leading 1 are zero.



# Sub problem 4

**Check if all other elements in a column, which contains a leading 1, are zero.**

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nonzero element above leading 1 in Row 2

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

not in ref